Matlab - ne namesti nekega syntactic toolbox?

- uvod v num. računanje
- sistem lin. enačb

$$\underset{?}{Ax=b}, \quad A \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$$

$$n \approx 10^6, \dots$$

- nelin. enačbe in sistemi
- lin. problem najmanjših kvadratov

$$Ax=b, \quad A \in \mathbb{R}^{m\times n}; \quad m \gg n$$

↳ predoločen sistem

- singularni razcep

PREMIČNA PIKA

$$x = \pm m \cdot b^e$$

$m = 0.c_1c_2\dots c_t$ mantisa

b ... baza (2 ali 10, 16)

e ... eksponent $L \leq e \leq U$

c_i ... št. med 0 in $b-1$

$c_1 \neq 0 \Rightarrow$ št. je normalizirano

t ... dolžina mantise

Primer: Računanje členov zaporedja:

i) $a_{n+1} - \frac{5}{2}a_n + a_{n-1} = 0, \quad a_0 = 1, \; a_1 = \frac{1}{2}$

ii) $a_{n+1} - \frac{10}{3}a_n + a_{n-1} = 0, \quad a_0 = 1, \; a_1 = \frac{1}{3}$

$$a_{n+1} - \frac{5}{2}a_n + a_{n-1} = 0$$

$$\lambda^{n+1} - \frac{5}{2}\lambda^n + \lambda^{n-1} = 0 \quad / \lambda^{n-1}$$

$$\lambda^2 - \frac{5}{2}\lambda + 1 = 0 \quad \Rightarrow \quad (\lambda-2)(\lambda-\tfrac{1}{2}) = 0$$

$$\lambda_1 = 1/2$$

$$\lambda_2 = 2$$

$$\Rightarrow a_n = A\lambda_1^n + B\lambda_2^n$$

$$a_0 = 1 = A + B \qquad B = 1 - A$$

$$a_1 = \frac{1}{2} = A\cdot\frac{1}{2} + 2B$$

$$\Rightarrow A = 1, \; B = 0$$

$$\Rightarrow a_n = \left(\tfrac{1}{2}\right)^n = 2^{-n}$$

ii) se v Matlabu ne izračuna pravilno, zakaj:

$$a_{n+1} - \frac{10}{3}a_n + a_{n-1} = 0, \quad a_0 = 1, \; a_1 = \frac{1}{3} + \varepsilon; \quad \varepsilon \neq 0$$

$$a_n = A\left(\tfrac{1}{3}\right)^n + B\,3^n$$

$$a_0 = 1 = A + B$$
$$a_1 = \frac{1}{3} + \varepsilon = \frac{1}{3}A + 3B$$

$$3\varepsilon = 8B$$
$$B = \frac{3\varepsilon}{8}, \quad A = 1 - B = \frac{8-3\varepsilon}{8}$$

$$a_n = \frac{8-3\varepsilon}{8} \cdot 3^{-n} + \frac{3\varepsilon}{8} 3^n$$

za večje n se napaka vedno bolj pozna

da to izboljšamo gremo v drugo smer in potem normaliziramo

$a_{99} = 1$, $a_{98} = 1$

$\rightarrow a$

2. pred. – 17.10.2025
↳ glej slajde

1.3 VRSTE NAPAK PRI NUMERIČNEM RAČUNANJU

vrednost y fun. $f: X \to Y$ pri danem x

→ dobimo približek $\hat{y}$ za y

$D = y - \hat{y}$ je celotna napaka približka

→ izvori napake

izračun $\sin \frac{x}{10}$ v $P(10, 4, \quad)$

- $x \to \bar{x}$ (predstavljivo št.)
- metoda za izračun $(\sin \frac{x}{10})$

  Taylorjeva vrsta:
  $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

  $$\sin x \doteq x - \frac{x^3}{3!}$$

  napaka ...

1.4 OBČUTLJIVOST PROBLEMA

→ premice (pravokotne, skoraj vzporedne)

→ Wilkinsonov zgled

iskanje lastnih vrednosti

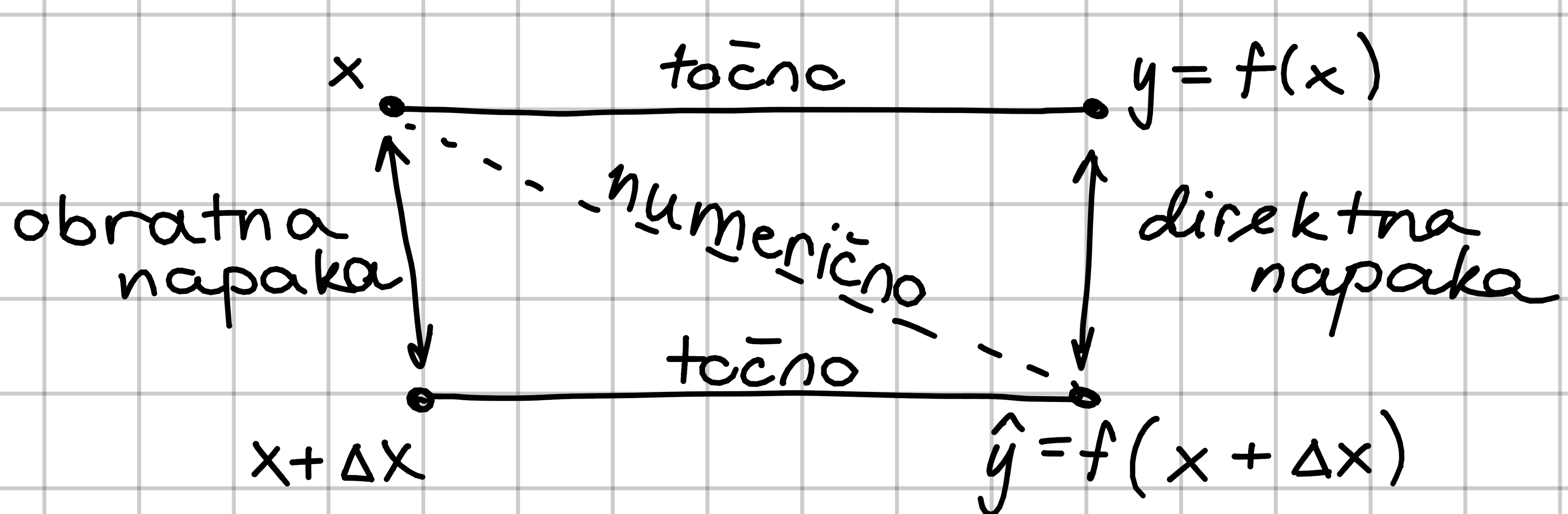
↓

če računaš lastne vrednosti kot ničle karakterističnega polinoma, so hitro velike napake

Stopnja občutljivosti

$$|f(x+\delta x) - f(x)| \approx |f'(x)| \cdot |\delta x|$$

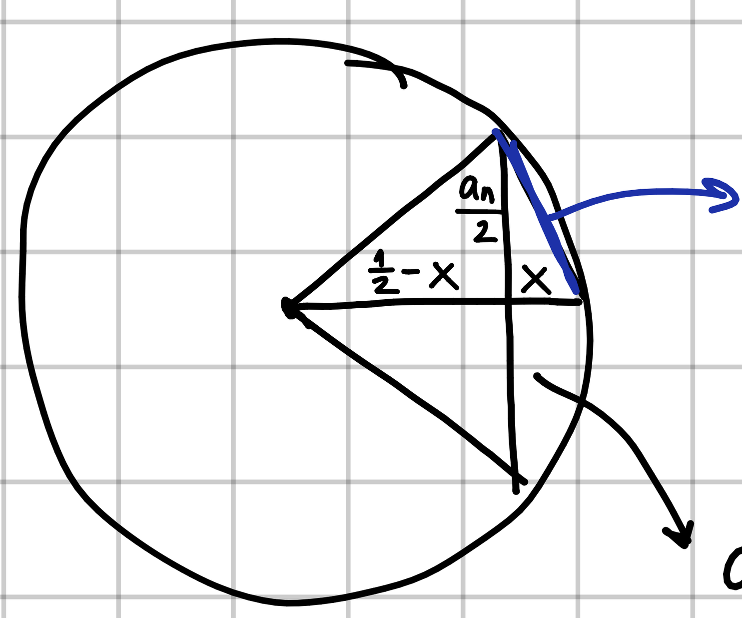
1.5 OBRATNA IN DIREKTNA STABILNOST



|direktna napaka| $\lesssim$ občutljivost $*$ |obratna napaka|

1.5.1 $\Rightarrow \; |\gamma| \lesssim 1 + nu + O(n^2)$
(to je na slajdih drugače)

Primer:



a_{2n} ... stranica pravilnega $2n$-kotnika

a_n ... stranica pravilnega n-kotnika

$r = \frac{1}{2}$

$$a_{2n} = \sqrt{\frac{1-\sqrt{1-a_n^2}}{2}}$$

$S_n = n \cdot a_n$

$S_{2n} = 2n \cdot a_{2n} \quad \Rightarrow \quad S_{4n} = 2n \sqrt{\frac{1-\sqrt{1-\left(\frac{S_n}{n}\right)^2}}{2}}$

$$S_6 = 3$$

$$\lim_{n\to\infty} S_{2n} = \pi$$

$$2n\sqrt{\frac{1-\sqrt{1-\left(\frac{S_n}{n}\right)^2}}{2}} = 2n\sqrt{\frac{\left(1-\sqrt{1-\left(\frac{S_n}{n}\right)^2}\right)\left(1+\sqrt{1-\left(\frac{S_n}{n}\right)^2}\right)}{2\left(1+\sqrt{1-\left(\frac{S_n}{n}\right)^2}\right)}} =$$

$$= 2n\sqrt{\frac{1-1+\left(\frac{S_n}{n}\right)^2}{2\left(1+\sqrt{1-\left(\frac{S_n}{n}\right)^2}\right)}} = S_n\sqrt{\frac{2}{1+\sqrt{1-\left(\frac{S_n}{n}\right)^2}}}$$

NUMERIČNO REŠEVANJE NELINEARNIH ENAČB

Ena enačba z eno neznanko

$f: I \subseteq \mathbb{R} \to \mathbb{R}$ ... iščemo rešitev $f(x)=0$
(zvezna) f nelinearna

ničla funkcije f (koren) ali rešitev enačbe $f(x)=0$

1.) $f(x) = ax^2+bx+c$; $a,b,c \in \mathbb{R}$
2.) $p_n(x) = 0$; p_n polinom stopnje n
3.) $f(x) = e^{-x} - x = 0 \iff e^{-x} = x$

$f(0) = e^0 - 0 = 1$

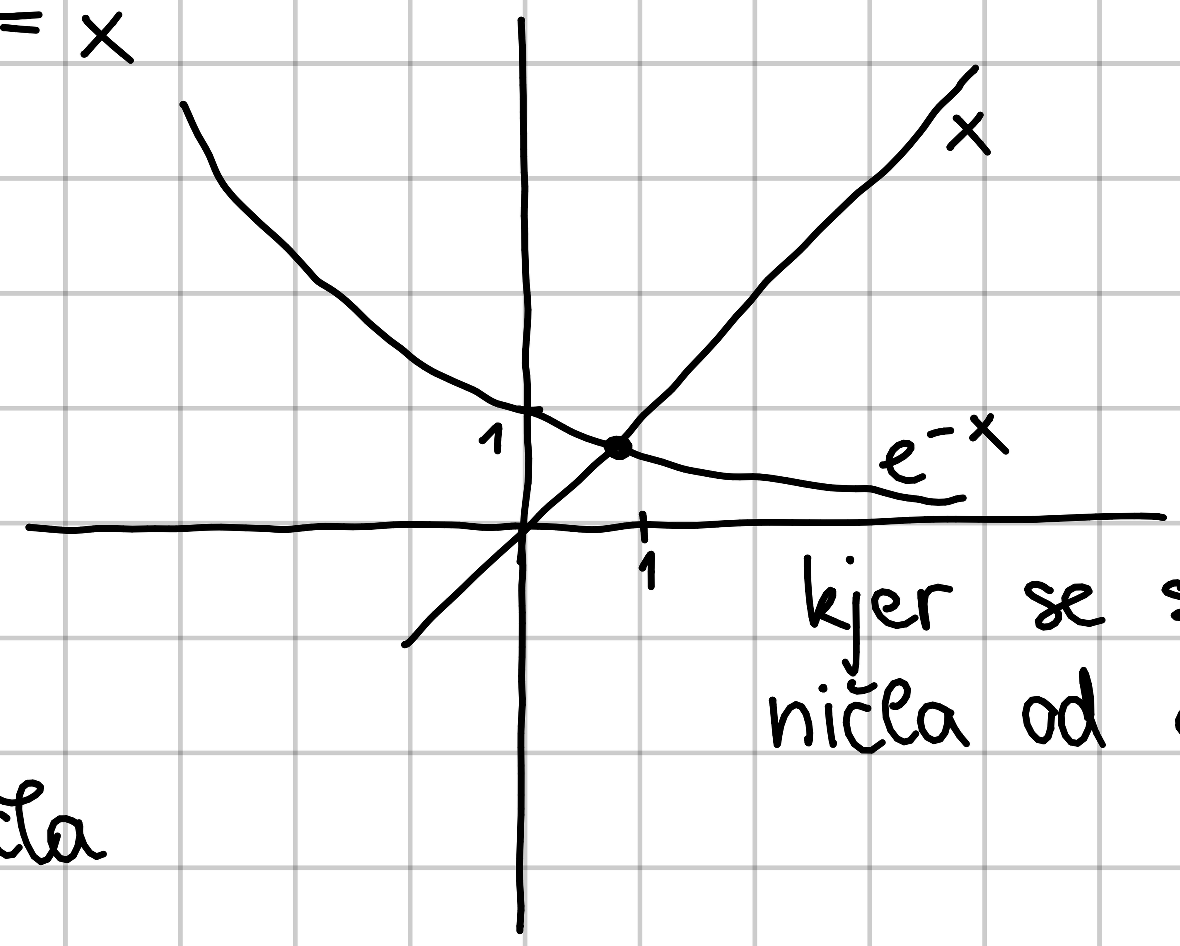
$f(1) = \frac{1}{e} - 1 < 0$

f zvezna $\Rightarrow$ na $[0,1]$ vsaj 1 ničla

$f'(x) = -e^{-x} - 1 < 0 \quad \forall x$

$\Rightarrow$ f je monotono padajoča

$\Rightarrow$ je samo 1 ničla



kjer se sekata, je ničla od $e^{-x} - x$

4.) $f(x) = tg\,x - x$

DEF.: Naj bo f m-krat zv. odvedljiva funkcija v okolici ničle $\alpha \in \mathbb{R}$.
i) α je enostavna ničla, če je $f'(\alpha) \neq 0$
ii) α je ničla stopnje m, če je $f^{(i)}(\alpha) = 0$, $i=0,1,\dots,m-1$
$f^{(m)}(\alpha) \neq 0$

Občutljivost ničle:

$$\varepsilon = f(\hat{\alpha}) = f(\alpha) + f'(\alpha)(\hat{\alpha}-\alpha) + \frac{f''(\alpha)}{2!}(\hat{\alpha}-\alpha)^2 + \dots + \frac{f^{(m)}(\alpha)}{m!}(\hat{\alpha}-\alpha)^m + \dots$$

enostavna ničla: $|\hat{\alpha}-\alpha| \doteq \frac{\varepsilon}{|f'(\alpha)|}$

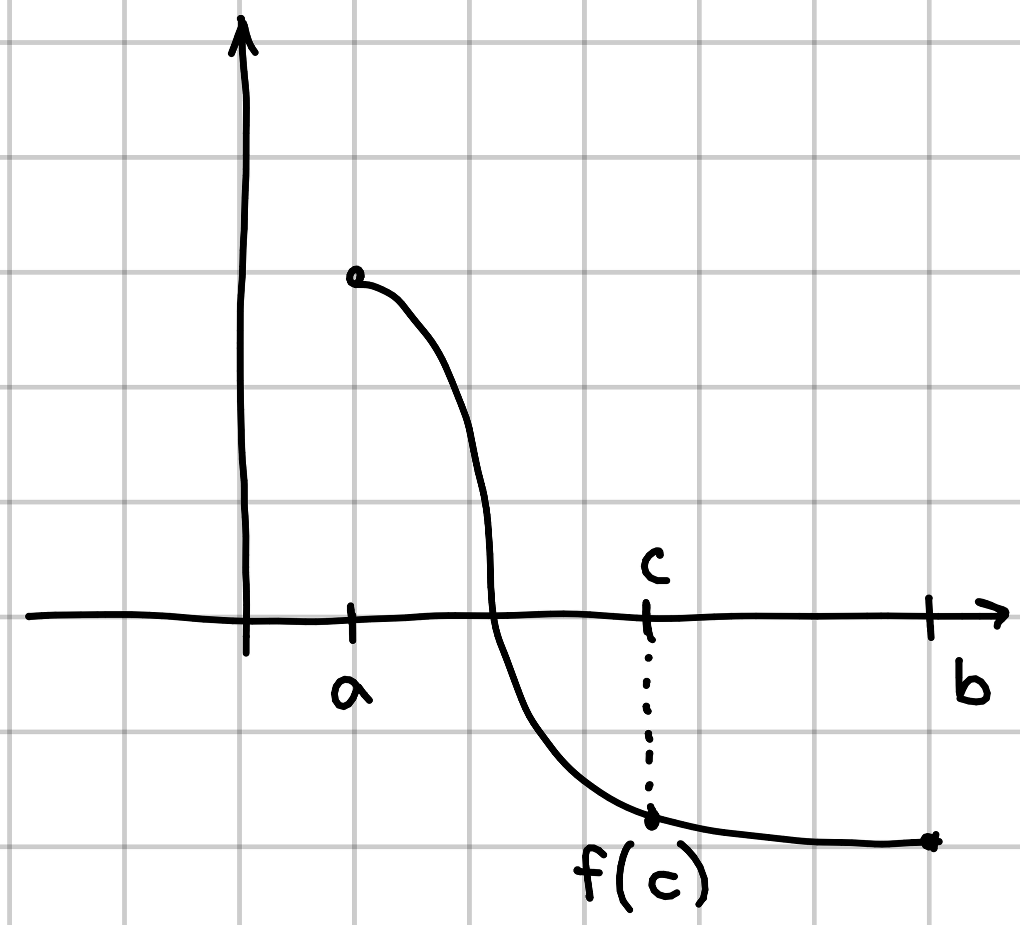
↳ toliko gremo lahko stran od ničle

$\Rightarrow$ občutljivost je $\frac{1}{|f'(\alpha)|}$

m-kratna ničla: $|\hat{\alpha}-\alpha| \doteq \sqrt[m]{\frac{\varepsilon\, m!}{|f^{(m)}(\alpha)|}}$

BISEKCIJA

$f: I \subseteq \mathbb{R} \to \mathbb{R}$, zvezna ; $I = [a,b]$, $f(a)f(b) < 0$



→ $f(a) \cdot f(c) < 0$

Ideja algoritma: $c = \frac{a+b}{2}$, $f(c)$

1) $f(a)f(c) < 0 \Rightarrow b = c$
2) $f(b)f(c) < 0 \Rightarrow a = c$

→ kjer je <0, pomeni, da je ničla med njima

Končamo: $|b-a| < \varepsilon$

Koliko korakov je potrebnih za določeno natančnost?

$$\frac{|b-a|}{2^k} < \varepsilon$$

na vsakem koraku razpolovimo interval

$$2^k > \frac{|b-a|}{\varepsilon} \quad / \log_2$$

$$k > \log_2 \left| \frac{b-a}{\varepsilon} \right|$$

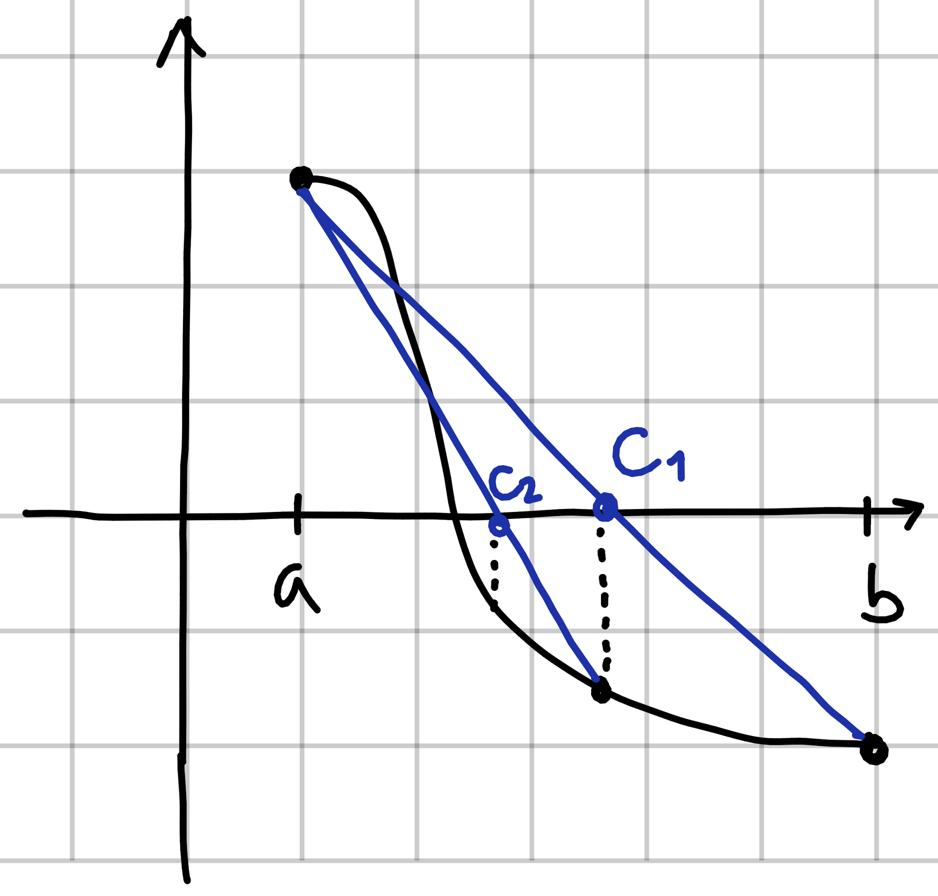
k je število korakov

Primer: $|b-a| = 1$, $\varepsilon = 10^{-10}$ $\qquad 3 < \log_2 10 < 4$

$$k > \log_2 \frac{1}{10^{-10}} = 0 - \log_2 10^{-10} = 10 \log_2 10 \doteq 30$$

→ dobra stvar bisekcije, da je ničla vedno znotraj intervala $[a,b]$
→ slaba: težko se posploši v več dimenzij oz. se ne da

METODA REGULA FALSI



→ ničla še vedno znotraj intervala

NAVADNA ITERACIJA

$f: I \subseteq \mathbb{R} \to \mathbb{R}$, $f(\alpha) = 0$, $\alpha \in I$

$$f(x) = 0 \overset{\text{ekv.}}{\iff} x = g(x)$$

$f(\alpha) = 0 \iff \alpha = g(\alpha)$; α je negibna točka (fiksna)

$x^2 + x + 1 = 0$

$\rightarrow x = -x^2 - 1$

$x = \pm\sqrt{-x-1}$

Standardne prevedbe:

1) $g(x) = x + f(x)$ $\quad (x + f(x) = x \iff f(x) = 0)$
2) $g(x) = x + C f(x)$; $C \neq 0$
3) $g(x) = x + h(x) f(x)$; $h(\alpha) \neq 0$, $(h(x) \neq 0)$

Generiramo zaporedje:

1) $x_0 \in \mathbb{R}$
2) $x_{r+1} = g(x_r)$, $r = 0, 1, \dots$

$\{x_r\}_{r=0}^{\infty}$

Recimo, da je: $\lim\limits_{r\to\infty} x_r = L$ (torej zap. konvergira)

$$\lim_{r\to\infty} x = \lim_{r\to\infty} g(x)$$

↳ zvezna

$$L = g\left(\lim_{r\to\infty} x\right) = g(L) \Rightarrow L = \alpha$$

Primer: $f(x) = x^3 - 5x + 1 = 0$ $\leftarrow$ 1 ali 3 realne ničle

1) $x = \dfrac{1+x^3}{5} = g_1(x)$

2) $x = \sqrt[3]{5x-1} = g_2(x)$

3. pred. - 23.10.2025

⋮

Matlab: $p = [1\ 0\ -5\ 1]$

$\rightarrow$ roots(p) ... fun., ki najde ničle za dani polinom

$x = \text{linspace}(\underbrace{-3, 3}_{\text{intv.}}, \underset{\downarrow}{100})$ št. točk

$y = \text{polyval}(p, x)$

2. možnost

f=@(x)x.^3 - 5*x + 1)

clf → pobriše, kar je od prej
hold on → riše na isto okno skoz
grid on → da imamo mrežo

$x_{r+1} = g_1(x_r)$; $r = 0, 1, \dots$
x_0 izberemo

→ 2. ničla je privlačna točka, drugi dve sta odbojni

⇒ je važno kakšen g izberemo

Princip skrčitve

$f: M \to M$ (M metrični prostor z razdaljo d)

Če $\exists m: 0 < m < 1$, da je $d(f(x), f(y)) \leq m \cdot d(x,y)$ $\forall x,y \in M$ potem je f skrčitev.

Vsaka skrčitev ima fiksno točko, če je M poln metrični prostor.

IZREK: Naj bo α negibna točka funkcije g in naj g na intervalu $I = [\alpha - d, \alpha + d]$, $d > 0$, zadošča Lipschitzovemu pogoju $|g(x) - g(y)| \leq m|x - y|$, za neko konst. m, $0 \leq m < 1$ in $\forall x,y \in I$.
Potem $\forall x_0 \in I$ zaporedje $x_{r+1} = g(x_r)$, $r = 0, 1, \dots$, konvergira k negibni točki α.

Velja ocena $|x_r - \alpha| \leq \frac{m}{1-m} |x_r - x_{r-1}|$

Dokaz: $x_r \in I$ za $\forall r$:

$x_0 \in I$ ✓
→ $x_{r+1} = g(x_r)$

$$|x_{r+1} - \alpha| = |g(x_r) - g(\alpha)| \leq m|x_r - \alpha| \leq m \cdot d < d \quad ✓$$

$$|x_r - \alpha| = |g(x_{r-1}) - g(\alpha)| \leq m|x_{r-1} - \alpha| \dots \leq m^r |x_0 - \alpha|$$

$$\lim_{r\to\infty} |x_r - \alpha| \leq \lim_{r\to\infty} m^r |x_0 - \alpha| = 0$$

$$\lim_{r\to\infty} x_r = \alpha$$

$$|x_{r+k} - x_r| = |x_{r+k} - x_{r+k-1} + x_{r+k-1} - x_{r+k-2} + x_{r+k-2} \cdots x_{r+1} - x_r|$$

$$\leq |x_{r+k} - x_{r+k-1}| + |x_{r+k-1} - x_{r+k-2}| + \cdots + |x_{r+1} - x_r|$$

$$\leq (m^k + m^{k-1} + m^{k-2} + \cdots + m) \cdot |x_r - x_{r-1}|$$
$$= m(m^{k-1} + m^{k-2} + \cdots + 1) \cdot |x_r - x_{r-1}|$$
$$= m\left(\frac{m^k - 1}{m-1}\right) \cdot |x_r - x_{r-1}|$$

$$\lim_{k\to\infty} |x_{r+k} - x_r| = |\alpha - x_r| \leq \lim_{k\to\infty} m\left(\frac{m^k-1}{m-1}\right)|x_r - x_{r-1}|$$
$$= \frac{m}{1-m}|x_r - x_{r-1}|$$

$$|x_{r+k} - x_{r+k-1}| = |g(x_{r+k-1}) - g(x_{r+k-2})|$$
$$\leq m|x_{r+k-1} - x_{r+k-2}|$$
$$\vdots$$
$$\leq m^k|x_r - x_{r-1}|$$

POSLEDICA: Naj bo iteracijska funkcija zv. odvedljiva v okolici negibne točke α in je $|g'(\alpha)| < 1$, potem obstaja interval I okrog α, da $\forall x_0 \in I$ zaporedje $x_{r+1} = g(x_r)$, $r = 0, 1, \ldots,$ konvg. k α.

Dokaz: Dovolj je pokazati, da je g skrčitev na I.

ker $|g'(\alpha)| < 1$ in g' zvezna, $\exists$ I okrog α, da je $|g'(x)| < 1$ $\forall x \in I$

Za poljubna $x, y \in I$ velja: $|g(x) - g(y)| = |g'(\xi)| \cdot |x - y|$
$\xi \in I$

↳ Lagrangeev izrek

če $m = \max_{\xi \in I} |g'(\xi)| < 1$ $\Rightarrow$ g je skrčitev

DEF.: Negibna točka α funkcije g je privlačna, če je $|g'(\alpha)| < 1$ in je odbojna, če $|g'(\alpha)| > 1$.

Primer: Če je α odbojna točka iteracije g, je nujno privlačna točka iteracije g^{-1} (g je odvedljiva) !

$$x = g(x) \iff x = g^{-1}(x)$$

$$|g'(\alpha)| > 1$$

$$(g^{-1}(x))' = \frac{1}{g'(g^{-1}(x))}$$

$$|(g^{-1}(\alpha))'| = \left|\frac{1}{g'(g^{-1}(\alpha))}\right| = \left|\frac{1}{g'(\alpha)}\right|$$

$$x = \operatorname{tg} x \Rightarrow x = \operatorname{arctg} x$$

Primer: $a > 0$, $g(x) = \frac{x^2 + a}{2x}$

$$x_{r+1} = g(x_r), \quad r = 0, 1, \dots$$

kam (če) konvg. $L = \lim_{r\to\infty} x_r$: $L = \frac{L^2 + a}{2L}$

$$2L^2 - L^2 = a$$
$$L^2 = a$$
$$L = \pm\sqrt{a}$$

$x_0 > 0$: to konvg. (če) k $+\sqrt{a}$

metoda za računanje korena : Babilonska metoda

Kako računati $\sqrt[k]{a}$?

→ Red konvergence

DEF.: Naj bo α privlačna negibna točka fun. g in $x_{r+1} = g(x_r)$, $r = 0, 1, \dots$
Pravimo, da je red konvergence (hitrost konvg.) enak $p > 0$, če $\exists$ konstanti $c_1, c_2 > 0$, da za vse dovolj pozne člene zaporedja velja

$$c_1 |x_r - \alpha|^p \leq |x_{r+1} - \alpha| \leq c_2 |x_r - \alpha|^p$$

x_r ima k točnih decimalk α → $\approx (10^{-k})^p = 10^{-pk}$

IZREK: Če je g v okolici negibne točke α <u>p-krat zvezno odvedljiva</u> in naj velja:

1) $\alpha = g(\alpha)$
2) $g^{(k)}(\alpha) = 0 \quad , k = 1, 2, \ldots, p-1$
3) $g^{(p)}(\alpha) \neq 0$

Potem je red konvergence p.

Dokaz:

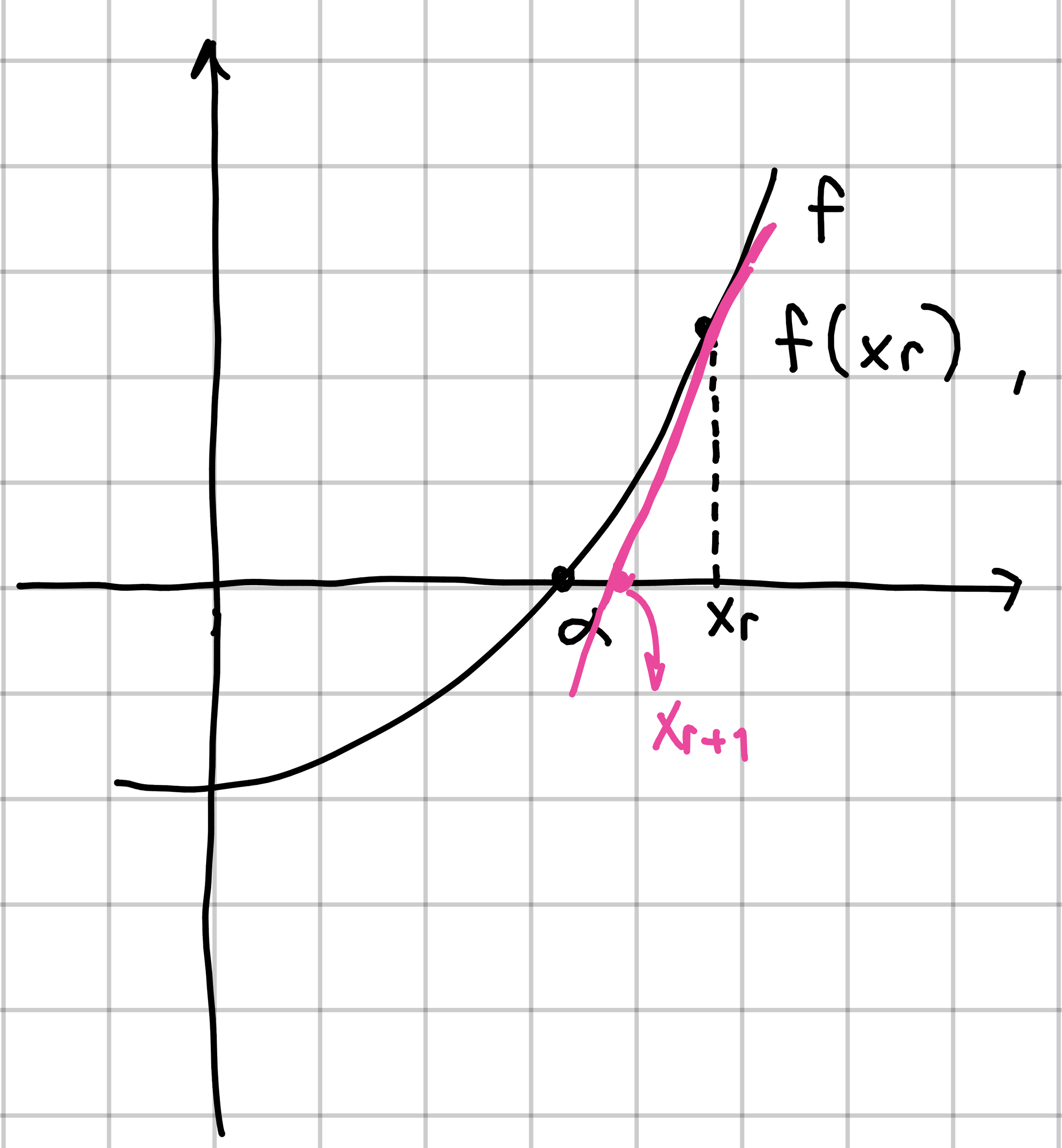
$$x_{r+1} = g(x_r) = g(x_r - \alpha + \alpha) =$$

$$= g(\alpha) + g'(\alpha)(x_r - \alpha) + \frac{g''(\alpha)}{2!}(x_r - \alpha)^2 + \ldots + \frac{g^{(p-1)}(\alpha)}{(p-1)!}(x_r - \alpha)^{p-1} + \underset{\neq 0}{\frac{g^{(p)}(\alpha)}{p!}}(x_r - \alpha)^p + \ldots$$

$$x_{r+1} - \alpha = \frac{g^{(p)}(\alpha)}{p!}(x_r - \alpha)^p + \ldots$$

$$|x_{r+1} - \alpha| = \left|\frac{g^{(p)}(\alpha)}{p!}\right| \cdot |x_r - \alpha|^p + \ldots$$

TANGENTNA (NEWTONOVA) METODA



$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Analitična izpeljava:

x_r približek za α

$$f(x_r + \Delta x_r) = 0$$

Taylor: $\quad f(x_r) + f'(x_r)\overset{x_{r+1} - x_r}{\overset{\parallel}{\Delta x_r}} + \ldots = 0$

$$x_{r+1} - x_r = -\frac{f(x_r)}{f'(x_r)}$$

$$x_{r+1} = x_r + \frac{f(x_r)}{f'(x_r)}$$

$$g(\alpha) = \alpha - \frac{f(\alpha)}{f'(\alpha)}$$

1) α je enostavna ničla

$f'(\alpha) \neq 0$

$$g(\alpha) = \alpha - \frac{0}{f'(\alpha)} = \alpha$$

$\Rightarrow \alpha$ je negibna točka

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} = \frac{f(x)f''(x)}{f'(x)^2}$$

$$\Rightarrow g'(\alpha) = 0$$

Primer: $\sqrt[k]{a} = ?$

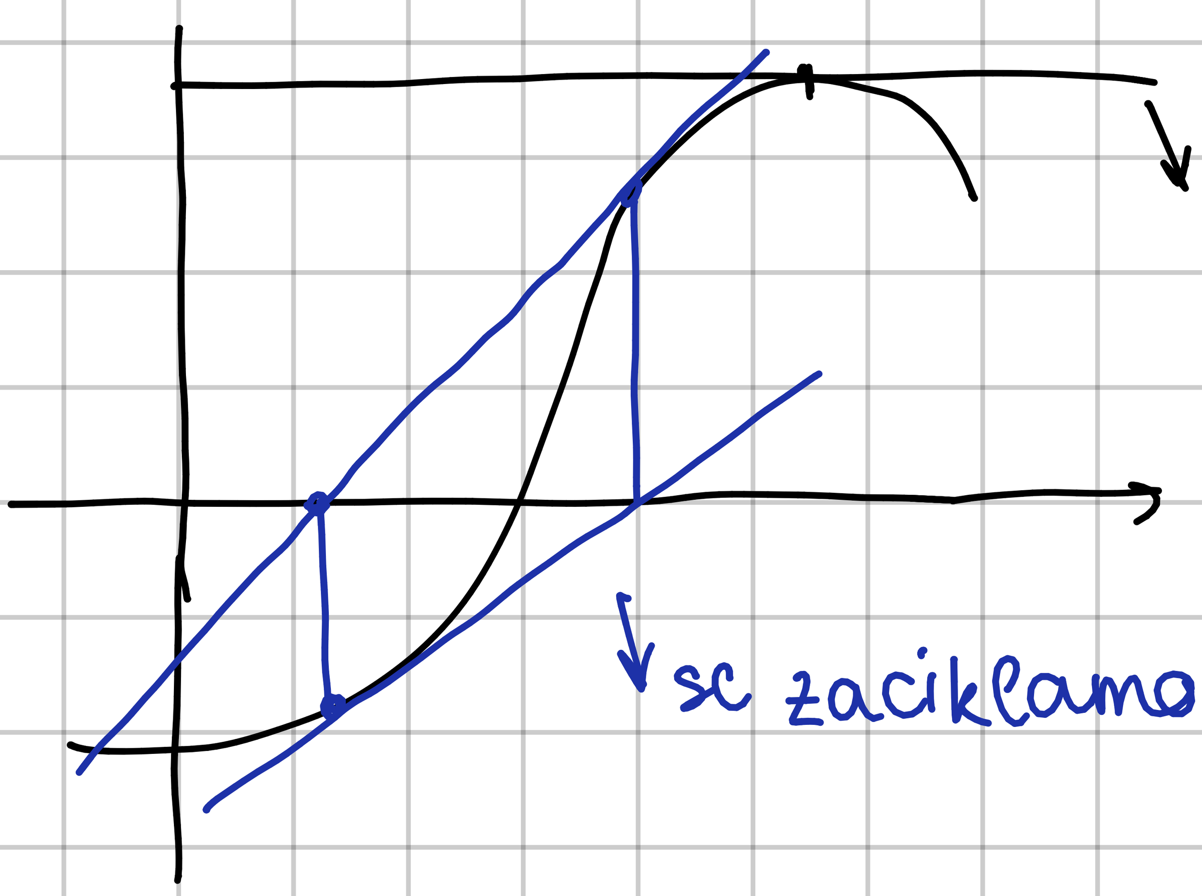
$$f(x) = x^k - a = 0$$

splošen način za računanje k-tega korena

$$f'(x) = kx^{k-1}$$

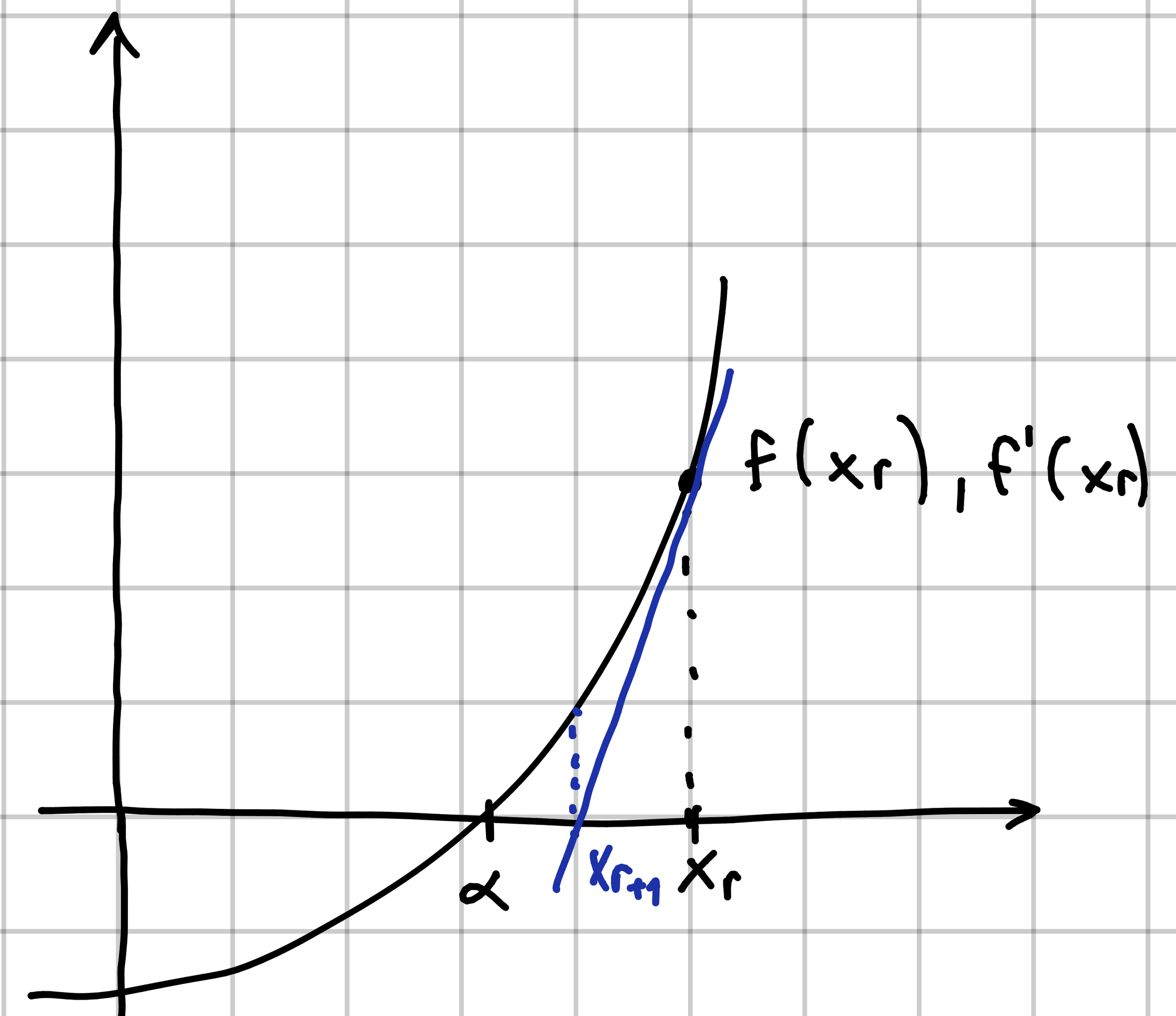
$$x_{r+1} = x_r - \frac{x_r^k - a}{k x_r^{k-1}} = \frac{(k-1)x_r^k + a}{k x_r^{k-1}}$$

Kdaj ne konvergira?



4.pred. – 24.10.2025

KVAZI-NEWTONOVA METODA



Namesto, da odvod računamo vsakič znova, uporabimo istega nekajkrat.

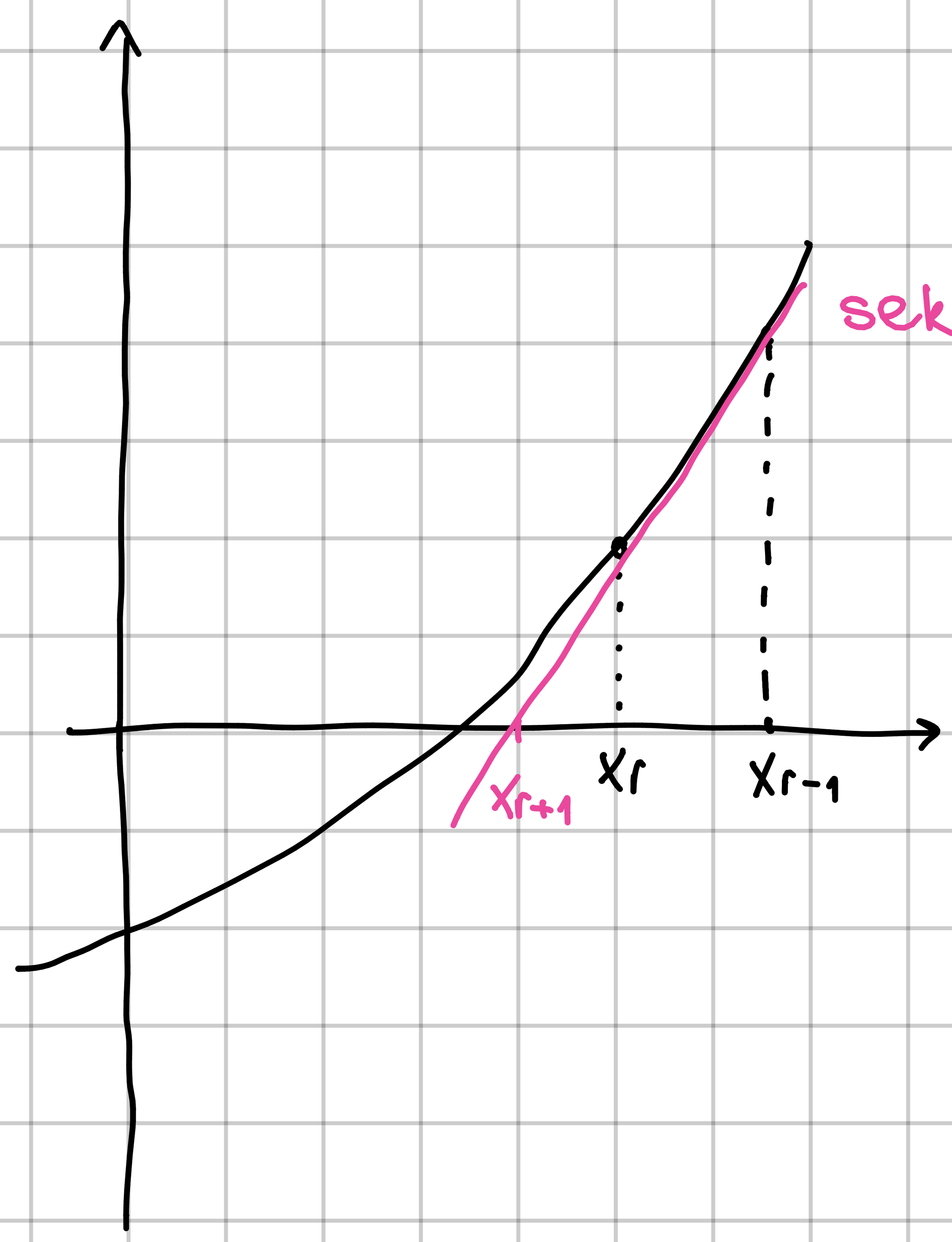
SEKANTNA METODA

$$x_{r+1} = x_r - \frac{f(x_r)}{f'(x_r)}$$ ... nov približek po tangentni metodi

$$f'(x_r) \doteq \frac{f(x_r) - f(x_{r-1})}{x_r - x_{r-1}}$$

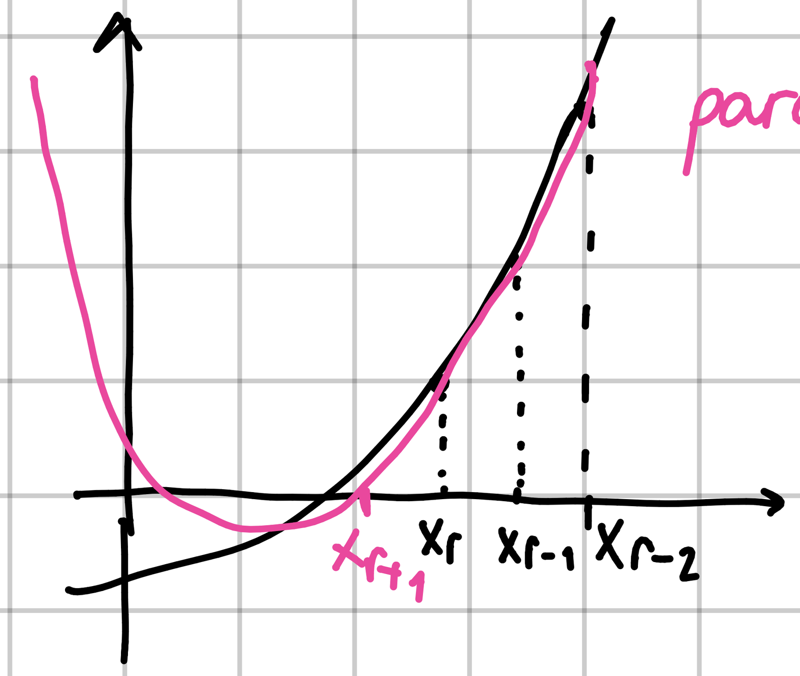
$$\Downarrow$$

$$x_{r+1} = x_r - \frac{f(x_r)(x_r - x_{r-1})}{f(x_r) - f(x_{r-1})}$$ ... sekantna metoda



Velja: red konvergence je $\frac{\sqrt{5}+1}{2} \doteq 1,6$
(superlinearna konvg.)

- Müllerjeva metoda
  → nov približek konstruiramo na podlagi treh prejšnjih



- Metoda f, f', f''
  → limitni primer Müllerjeve metode: zadnje 3 približke „stisnemo" v 1 točko

- Halleyeva metoda
  → metoda je bila za določiti tirnico Halleyevega kometa

- Brentova metoda (fzero : MATLAB)
  → kombinacija bisekcije, sekantne metode in inverzne interpolacije (kvadratične)

x	x_0	x_1	$\dots$	x_n
$f(x)=y$	f_0	f_1	$\dots$	f_n

določimo interp. polinom
$p_n(x_i) = f_i \quad \forall i = 0, 1, \dots, n$
če v tabeli zamenjamo vrstici, interpoliramo inverzno fun.
$f^{-1} : f^{-1}(0) = \alpha$

RAČUNANJE NIČEL POLINOMA

→ iščemo lastne vrednosti neke matrike (ničle karakterističnega polinoma)

- Pridružena matrika:

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \qquad ; a_n \neq 0$$

$$\tilde{p}_n(x) = x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \dots + \frac{a_1}{a_n} x + \frac{a_0}{a_n}$$

↳ monični polinom

$$C_{\tilde{p}_n} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0/a_n & -a_1/a_n & \dots & \dots & -\frac{a_{n-1}}{a_n} \end{bmatrix}$$

↳ karak. polinom te matrike se za konst. faktor razlikuje od $\tilde{p}_n$

$$\det(C_{\tilde{p}_n} - \lambda I) = K \cdot \tilde{p}_n(\lambda)$$

Torej so ničle $\tilde{p}_n$ ravno lastne vrednosti matrike $C_{\tilde{p}_n}$

MATLAB: roots

$$f(x) = e^{-x} - x = 0$$
$$e^{-x} = x$$
fzero(f, zač. približek)

- Durand-Kernerjeva metoda

p polinom st. n (monični)

$$p(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_n)$$

iščemo $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$

- recimo, da so $z_1, \ldots, z_n$ približki za $\alpha_1, \alpha_2, \ldots, \alpha_n$
- naj bodo Δz_i popravki : $z_i + \Delta z_i = \alpha_i$

Veljati mora: $p(z) = (z-(z_1+\Delta z_1))(z-(z_2+\Delta z_2))\cdots(z-(z_n+\Delta z_n))$

$$= \prod_{j=1}^{n}(z-z_j) - \sum_{j=1}^{n}\Delta z_j \prod_{\substack{k=1\\k\neq j}}^{n}(z-z_k) + \underbrace{\cancel{\sum_{\substack{j,k=1\\j<k}}^{n}\Delta z_j \Delta z_k \prod_{\substack{i=1\\i\neq j,k}}^{n}(z-z_i)}}_{\text{zanemarimo}}$$

Vstavimo $z = z_i$ v okrnjeni del :

$$\Delta z_i = -\frac{p(z_i)}{\prod_{\substack{k=1\\k\neq i}}^{n}(z_i - z_k)}$$

$$z_i^{(r+1)} = z_i^{(r)} - \frac{p(z_i^{(r)})}{\prod_{\substack{k=1\\k\neq i}}^{n}(z_i^{(r)} - z_k^{(r)})}, \quad i = 1, 2, \ldots, n$$

SISTEMI LINEARNIH ENAČB

→ Kako množimo matriko in vektor?

$A \in \mathbb{R}^{m \times n}$ predstavlja linearno preslikavo
$A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

1) $A(\lambda x) = \lambda A x$
2) $A(x+y) = Ax + Ay$

$$A = (a_{ij})_{i=1,j=1}^{m \quad n}$$

$A \in \mathbb{R}^{m \times n}$
$x \in \mathbb{R}^{n \times 1}$ (vektor je stolpec)

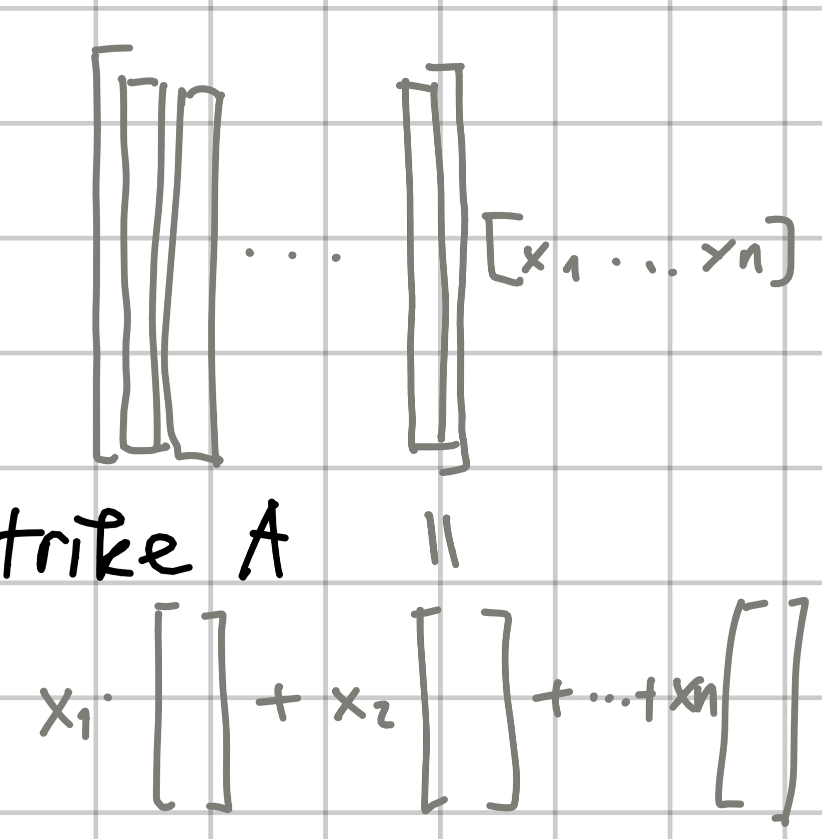
$$(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j$$

$A = [a_1, a_2, \dots, a_n]$ → seznam stolpcev

$a_i \dots$ i-ti stolpec matrike A

$x = [x_1, x_2, \dots, x_n]^T$

$$\Rightarrow \quad Ax = \sum_{j=1}^{n} x_j a_j$$



$Ax = b$ ima rešitev $\Leftrightarrow$ $\text{rang}\, A = \text{rang}([A, b])$

$(\Rightarrow)$: $Ax = b$, x rešitev:
$$b = \sum_{j=1}^{n} x_j a_j$$

$(\Leftarrow)$: $\text{rang}\, A = \text{rang}([A, b])$
$$b = \sum_{j=1}^{n} x_j a_j \quad \Leftrightarrow \quad Ax = b \text{ ima rešitev}$$

Primer:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 3 & 4 \end{bmatrix} \qquad x = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

$$Ax = \sum_{j=1}^{2} x_j a_j = 2 \cdot \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

$$\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n} = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n} = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn} = b_m \end{array}} \iff \begin{array}{l} Ax = b \\ A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ x \in \mathbb{R}^n \end{array}$$

- vektor $x = [x_1\ x_2\ \dots\ x_n]^T$
- matrika $A = [a_{ij}]_{i=1,\,j=1}^{m,\quad n} = [a_1\ a_2\ \dots\ a_n]$

$$A = \begin{bmatrix} \tilde{a}_1^T \\ \vdots \\ \tilde{a}_m^T \end{bmatrix}$$

- $e_j = [0\ 0 \dots 0\ 1\ 0 \dots 0]$

  $\uparrow$ j-to mesto

  $Ae_j = a_j$ (j-ti stolpec matrike A)

- $\operatorname{Im} A = \{Ax;\ x \in \mathbb{R}^n\} = \mathcal{L}in\{a_1, a_2, \dots, a_n\}$
- $\operatorname{rang} A = \dim \operatorname{Im} A$ ... št. lin. neodvisnih stolpcev matrike A
- $\ker A = \{x \in \mathbb{R}^n;\ Ax = 0\}$
- $\operatorname{rang} A + \dim \ker A = n$
- $A \in \mathbb{R}^{n\times n}$ ... kvadratna matrika

  A je obrnljiva (nesingularna):
  1) $\exists A^{-1} : \mathbb{R}^n \to \mathbb{R}^n$
  2) $\det A \neq 0$
  3) $\operatorname{rang} A = n$
  4) ne $\exists$ $x \in \mathbb{R}^n, x \neq 0 ; Ax = 0$

  A je simetrična, če $A^T = A$

  A je hermitska, če $A^H = A$
  ↳ transponirana in konjugirana

  A je ortogonalna, če $A^T A = A A^T = I$
  $\rightsquigarrow A^T = A^{-1}$

Vektorske in matrične norme

DEF.: Vektorska norma je preslikava $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}_+$, za katero velja:

1) $\|x\| \geq 0$; $\|x\| = 0 \iff x = 0$
2) $\|\lambda x\| = |\lambda| \cdot \|x\|$
3) $\|x+y\| \leq \|x\| + \|y\|$

Primeri vektorskih norm:

1) $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$ ... prva norma
2) $\|x\|_2 = \sqrt{\sum_{j=1}^{n} |x_j|^2}$ ... druga/evklidska norma
3) $\|x\|_p = \sqrt{\sum_{j=1}^{n} |x_j|^p}$, $p \in \mathbb{N}$
4) $\|x\|_\infty = \max_{1 \leq j \leq m} |x_j|$

$d(x,y) = \|x-y\|$

$C_2\|x\| \leq \|x\|' \leq C_1\|x\|$ na končnodim. prostorih

DEF.: Matrična norma je preslikava $\|\cdot\| : \mathbb{R}^{m \times n} \to \mathbb{R}_+$, da velja:

1) $\|A\| \geq 0$, $\|A\| = 0 \iff A = 0$
2) $\|\lambda A\| = |\lambda| \|A\|$
3) $\|A+B\| \leq \|A\| + \|B\|$
4) submultiplikativnost: $\|AB\| \leq \|A\| \cdot \|B\|$

Najbolj znane matrične forme:

Frobeniusova norma:

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$

↳ ni operatorska

Operatorske norme:

Definiramo jih s pomočjo vektorskih norm:

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

Lema: Vsaka operatorska norma je res matrična norma.

1) $\|A\| \geq 0$ ✓

$\|A\| = 0 \iff A = 0$

$(\Leftarrow)$: $A = 0 \Rightarrow \|Ax\|_v = 0 \ \forall x$ ✓

$(\Rightarrow)$: $\|A\|=0 \Rightarrow \max\limits_{x\neq 0} \frac{\|Ax\|_v}{\|x\|_v} = 0 \Rightarrow \frac{\|Ax\|_v}{\|x\|_v} = 0 \quad \forall x$

$\Rightarrow \|Ax\|_v = 0 \quad \forall x \Rightarrow Ax = 0 \ \forall x \Rightarrow A = 0$

2) $\|\lambda A\| = |\lambda| \|A\|$

$$\|\lambda A\| = \max_{x\neq 0} \frac{\|\lambda A x\|_v}{\|x\|_v} = \max_{x \neq 0} \frac{|\lambda| \|Ax\|_v}{\|x\|_v} =$$

$$= |\lambda| \cdot - \| - = |\lambda| \cdot \|A\|$$

3) $$\|A+B\| = \max_{x\neq 0} \frac{\|(A+B)x\|_v}{\|x\|_v} = \max \frac{\|Ax+Bx\|_v}{\|x\|_v}$$

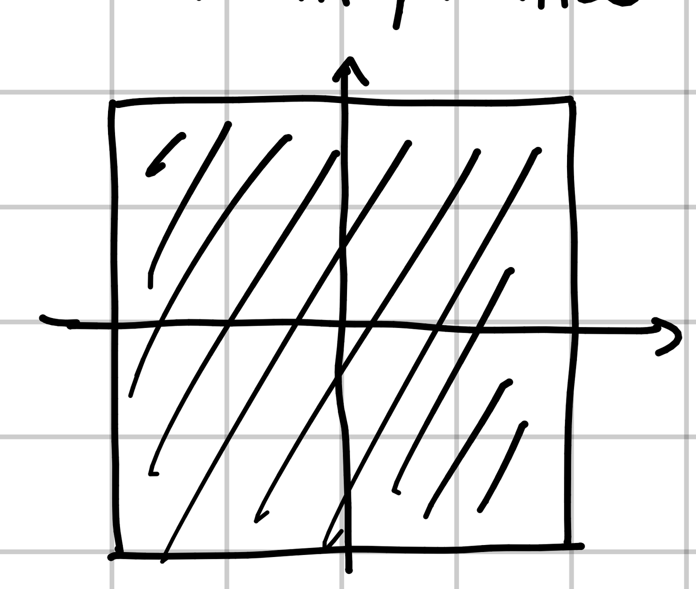
$$\leq \max \frac{\|Ax\| + \|Bx\|}{\|x\|} \leq \|A\| + \|B\|$$

7. 11. 2025

$\|x\|_1, \|x\|_\infty$

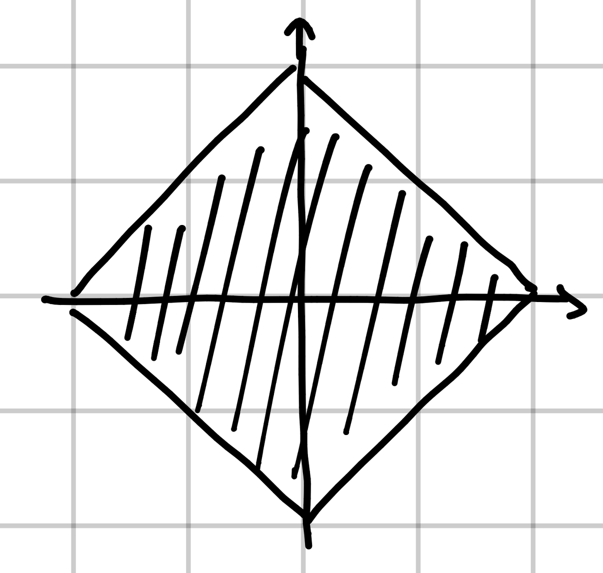
- $$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{\|x\|_\infty = 1} \|Ax\|_\infty$$ enotska krogla

  $$= \max_{1\leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$$ → vsote po vrsticah



- $$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$$ → vsote po stolpcih

  enotska krogla



- $$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \max_{\lambda_i} \sqrt{\lambda_i(A^TA)} = \sqrt{\rho(A^TA)}$$ ρ... spektralni radij

  ↳ $\lambda_i(A^TA)$ i-ta lastna vrednost matrike A^TA

Lema: $$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

Dokaz: Naj bo $x \in \mathbb{R}^n$; $\|x\|_\infty = 1$

obstaja $1 \leq k \leq n$: $|x_k| = 1$

$$Ax = \left(\sum_{j=1}^{n} a_{ij} x_j\right)_{i=1}^{m}$$ → vektor

$$\|Ax\|_\infty = \max_{1\leq i\leq m} \left|\sum_{j=1}^{n} a_{ij} x_j\right| \leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}||x_j|$$

$$\leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

$$\Rightarrow \|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty \leq \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

Dovolj je poiskati tak x, da $\|x\|_\infty = 1$ in $\|Ax\|_\infty = \max\limits_{1\leq i\leq m} \sum\limits_{j=1}^{n} |a_{ij}|$

zagotovo $\exists k: \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}| = \sum_{j=1}^{n} |a_{kj}|$

izberemo $x_j = \text{sign}(a_{kj})$, $j \in [n]$ $\Rightarrow \|x\|_\infty = 1$

$$\sum_{j=1}^{n} |a_{kj}| = \sum_{j=1}^{n} a_{kj} x_j = (Ax)_k$$

$$\Rightarrow \|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

Primer:

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ 1 & 1 & 1 \\ 3 & 4 & -5 \end{bmatrix}$$

$\|A\|_\infty = 12$ po vrsticah abs $\max(1+2+3, 2+1+4, 1+1+1, \overbrace{3+4+5}^{12})$

$\|A\|_1 = 13$ po stolpcih abs $\max(1+2+1+3, 2+1+1+4, \underbrace{3+4+1+5}_{13})$

$\|A\|_F = \sqrt{\Sigma(\,)^2} =$

$\|A\|_2 = \max \sqrt{\text{lastne vrednosti } A^T A}$ ... eig v MATLAB-u

norm(A, {1, 2, 'fro', inf})

OBČUTLJIVOST SISTEMA LIN. ENAČB

tega nisem predelala za kviz

$Ax = b$

$A \to A + \delta A$

napaka pri merjenju

$b \to b + \delta b$

$\Rightarrow x \to x + \delta x$

intuitivno: $\frac{\|\delta x\|}{\|x\|} \approx C_1 \frac{\|\delta A\|}{\|A\|} + C_2 \frac{\|\delta b\|}{\|b\|}$

relativna napaka

→ ni čisto tako

Lema: Če za matriko X velja $\|X\| < 1$ v matrični normi, v kateri je $\|I\| = 1$, potem je matrika $I - X$ obrnljiva in velja

$$\|(I - X)^{-1}\| \leq \frac{1}{1 - \|X\|}$$

Dokaz: Recimo, da je $I - X$ izrojena.

Potem $\exists z \neq 0 : (I - X)z = 0 \Rightarrow z = Xz \Rightarrow$

$\|z\| = \|Xz\| \leq \|X\| \cdot \|z\| \Rightarrow \|X\| \geq 1$ ↯

$\Rightarrow I - X$ je obrnljiva

Opazimo, da je $(I-X)^{-1} = \sum_{i=0}^{\infty} X^i$ (doma) $\qquad (1-x)^{-1} = \frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$

$$\|(I-X)^{-1}\| = \|\sum_{i=0}^{\infty} X^i\| \leq \underbrace{\sum_{i=0}^{\infty}\|X\|^i}_{= \|I\| + \|X\| + \|X\|^2 + \dots,\ \|I\| = 1} = \frac{1}{1-\|X\|}$$

$$(A+\delta A)(x+\delta x) = b + \delta b$$

$$\cancel{Ax} + A\delta x + \delta A x + \delta A \delta x = \cancel{b} + \delta b$$

$$(A+\delta A)\delta x = \delta b - \delta A x$$

$$\delta x = (A+\delta A)^{-1}(\delta b - \delta A x)$$

$$= (I + A^{-1}\delta A)^{-1} A^{-1}(-\delta A x + \delta b)$$

predpostavimo, da je $\|A^{-1}\| \cdot \|\delta A\| < 1$:
po prejšnji lemi je $I + A^{-1}\delta A$ obrnljiva in velja:

$$\|(I + A^{-1}\delta A)^{-1}\| \leq \frac{1}{1-\|A^{-1}\delta A\|} \leq \frac{1}{1-\|A^{-1}\|\,\|\delta A\|}$$

Ker je $\|\delta x\| \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\,\|\delta A\|}\left(\|\delta A\|\,\|x\| + \|\delta b\|\right)$, dobimo

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\cdot\|\delta A\|}\left(\frac{\|\delta A\|}{\|A\|}\|A\| + \frac{\|\delta b\|}{\|x\|}\cdot\frac{\|A\|}{\|A\|}\right) \leq$$

$$\leq \frac{\|A^{-1}\|\cdot\|A\|}{1-\|A^{-1}\|\cdot\frac{\|A\|}{\|A\|}\cdot\|\delta A\|}\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)$$

$$\frac{\|A^{-1}\|\cdot\|A\|}{1-\underbrace{\|A^{-1}\|\cdot\|A\|}_{\mathcal{K}(A)}\cdot\frac{\|\delta A\|}{\|A\|}}$$

$\|A^{-1}\|\cdot\|A\| := \mathcal{K}(A)$ ... št. občutljivosti ali pogojenostno št.
(condition number)

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\mathcal{K}(A)}{1-\mathcal{K}(A)\cdot\frac{\|\delta A\|}{\|A\|}}\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right)$$

vedno je $\mathcal{K}(A) \geq 1$:

$$\|I\| = 1 = \|AA^{-1}\| \leq \|A\|\,\|A^{-1}\| = \mathcal{K}(A)$$

Če je A ortogonalna $\Rightarrow A^{-1} = A^T$ je tudi ortogonalna

$$\Rightarrow \|A_2\| = 1 \text{ in } \|A^{-1}\|_2 = \|A^T\|_2 = 1 \Rightarrow \mathcal{K}_2(A) = 1$$

ponavadi velja: $\mathcal{H}(A) = 10^a \Rightarrow$ rešitev $Ax=b$ je točna $\approx 16-a$ mest

$\mathcal{H}$ v MATLABU je cond( )
$x = A \backslash b$

Hilbertova matrika

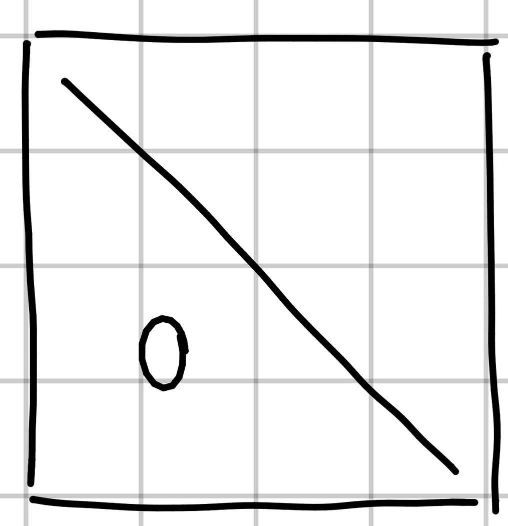
$$H_n = \left(\frac{1}{i+j-1}\right)_{i=1}^{n}$$

$\to$ veliki $\mathcal{H}$

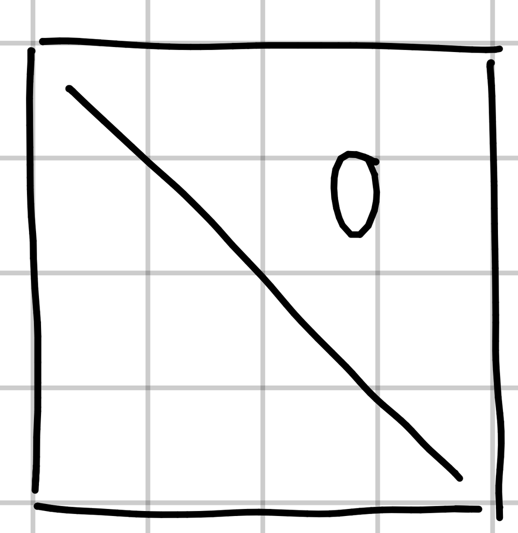
hilb(8) $\rightsquigarrow H_8$
v MATLABU

REŠEVANJE TRIKOTNIH SISTEMOV LIN. ENAČB

- A je zgornje trikotna, če je $a_{ij} = 0$ za $i > j$



- A je spodnje trikotna, če je $a_{ij} = 0$ za $i < j$



$\to$ Reševanje sistema, katerega matrika je spodnje trikotna

$Lx = b$;

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ l_{n1} & l_{n2} & & \dots & l_{nn} \end{bmatrix}$$

$$\begin{aligned} l_{11} \cdot x_1 + 0 \cdot \dots &= b_1 \\ l_{21} \cdot x_1 + l_{22} \cdot x_2 + \dots &= b_2 \\ \vdots \qquad & \quad \vdots \\ l_{i1} \cdot x_1 + \dots + l_{ii} x_i + \dots &= b_i \\ \vdots \qquad & \quad \vdots \\ l_{n1} x_1 + \dots + l_{nn} x_n &= b_n \end{aligned} \quad \Rightarrow x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}}$$

L je nesingularna $\Leftrightarrow l_{ii} \neq 0 \quad \forall i \in [n]$
ker je det produkt diagonalcev
$\Rightarrow$ det različna od 0

Koda:

i-ta vrstica od 1 do i-1

```
for i = 1:n
    x(i) = (b(i) - L(i, 1:i-1) * x(1:i-1)) / L(i,i) ;
end
```

PREMO ali DIREKTNO vstavljanje

→ prostor : $O(n^2)$
čas : $O(n^2)$

preštevanje operacij:

–	/	*	+
1	1	i-1	i-2

$2+2i-3 = 2i-1$ operacij za i-to komponento

$$\rightsquigarrow \sum_{i=1}^{n}(2i-1) = 2\cdot\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 = \cancel{2}\,\frac{n(n+1)}{\cancel{2}} - n = n^2+n-n = n^2$$

→ Reševanje sistema, katerega matrika je zgornje trikotna

$Ux=b$;

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

```
for i = n:-1:1
    x(i) = (b(i) - U(i, i+1:end) * x(i+1:end)) / U(i,i) ;
end
```

OBRATNO vstavljanje

→ prostor : $O(n^2)$
čas : $O(n^2)$ ← ustni izpit

–	/	*	+
1	1	n-i	n-i-1

$2n-2i+1$

$$2\sum_{i=1}^{n} n - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = 2n^2 - 2\cdot\frac{n(n+1)}{2} + n =$$
$$= 2n^2 - n^2 - n + n = n^2$$

14.11.2025

$Ax = b$; $A \in \mathbb{R}^{n\times n}$, $\det A \neq 0$

$x = A^{-1}b$ (tega numerično ne delamo)

LU RAZCEP

Ideja: $A = L\cdot U$; razcep matrike A

$$L = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{bmatrix}, \quad U = \begin{bmatrix} \diagdown & & * \\ & \diagdown & \\ 0 & & \diagdown \end{bmatrix}$$

$$\det A = \underset{\substack{\shortparallel \\ 1}}{\det L} \cdot \det U$$

$\det U \Rightarrow$ ne more biti 0 na diagonali

$$Ax = b$$
$$(LU)x = b$$
$$L(Ux) = b$$

$Ux = y$; nova neznanka

$\Rightarrow$ $Ly = b$ ... direktno vstavljanje ($O(n^2)$)

$Ux = y$ ... obratno vstavljanje ($O(n^2)$) $\Rightarrow x$

Ne $\exists$ vedno LU razcep (četudi je $\det A \neq 0$)

Primer:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \cdot \begin{bmatrix} b & c \\ 0 & d \end{bmatrix} = \begin{bmatrix} \overset{\overset{0}{\shortparallel}}{b} & c \\ \underset{\substack{\shortparallel \\ 0}}{ab} & ac+d \end{bmatrix}$$

$0 \longrightarrow \times$

Lema: Naj bo $A \in \mathbb{R}^{n\times n}$ nesingularna matrika. Če so vsi glavni minorji matrike A različni od 0, $\exists$ enolični LU razcep. Velja tudi obratno.

(glavni minorji: $\det(A(1:k, 1:k)) \neq 0$, $k = 1, \dots, n$)

Dokaz:

($\Leftarrow$): Denimo, da LU razcep $\exists$.

$$A = L\cdot U$$

$$\begin{bmatrix} A_k & \\ & \end{bmatrix} = \begin{bmatrix} L_k & 0 \\ * & \begin{smallmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{smallmatrix} \end{bmatrix} \cdot \begin{bmatrix} U_k & * \\ 0 & \ddots \end{bmatrix}$$

bločno množenje matrik

$$\Rightarrow A_k = L_k \cdot U_k$$
$$\det A_k = \det L_k \cdot \det U_k$$
$$= 1 \cdot \prod_{j=1}^{k} \underset{\substack{\neq \\ 0}}{u_{jj}} \neq 0 \qquad \forall k$$

$(\Rightarrow)$: Indukcija na n:

$n=1$: $A = [a_{11}] = \underset{L}{[1]} \cdot \underset{U}{[a_{11}]}$ ker A nesingularna, $u_{11} \neq 0$

$n \rightsquigarrow n+1$:

$$A = \left[\begin{array}{c|c} A_n & a \\ \hline b^T & \underbrace{\ast}_{a_{n+1,n+1}} \end{array}\right] = \left[\begin{array}{c|c} L_n & 0 \\ \hline \ell^T & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} U_n & u \\ \hline 0 & \alpha \end{array}\right]$$

vemo: v A_n vsi minorji različni od 0
$A_n = L_n \cdot U_n$ $\Leftarrow$ v A tudi

Preveriti moramo, da so ℓ, u in α enolično določeni.

$\ell^T \cdot U_n = b^T$

$\ell^T \cdot u + \alpha = a_{n+1,n+1}$

$L_n \cdot u = a$

$\Downarrow$

u (direktno vstavljanje)

$\ell^T U_n = b^T \quad /^T$

$U_n^T \ell = b$

$\Downarrow$

ℓ (direktno vstavljanje (U_n^T, b))

$\alpha = a_{n+1,n+1} - \ell^T u$ $\square$

Elementarne eliminacije

Primer: $x = [1, 2, \overset{\neq 0}{4}, -8, 6, 1]^T$

Iščemo preprosto matriko L_3 tako, da bo imel vektor $L_3 x$ ničle na mestih 4,5,6 ter element na mestu 3 $\neq 0$.

$$\ell_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -8/4 \\ 6/4 \\ 1/4 \end{bmatrix} \qquad L_3 = I_6 - \ell_3 e_3^T = I_6 - \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 3/2 \\ 1/4 \end{bmatrix} [0 \ 0 \ 1 \ 0 \ 0 \ 0]$$

$$= I_6 - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 3/2 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -3/2 & 0 & 1 & 0 \\ 0 & 0 & -1/4 & 0 & 0 & 1 \end{bmatrix}$$

$$L_3 x = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L_1 \begin{bmatrix} * & \\ * & \\ \vdots & * \\ \vdots & \\ * & \end{bmatrix} = \begin{bmatrix} * & \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix}$$

Splošno: $x = [x_1, \dots, x_k, x_{k+1}, \dots, x_n]^T$; $x_k \neq 0$

Iščemo ℓ_k tako, da je $L_k x = (I_n - \ell_k \cdot e_k^T)x =$

$$= [*, *, \dots, *, 0, \dots, 0]^T$$

$$\ell_k = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_{k+1}/x_k \\ \vdots \\ x_n/x_k \end{bmatrix} \qquad L_k = I_n - \ell_k \cdot e_k^T$$

doma: $L_k x = [*, \dots, *, 0, \dots, 0]^T$

Nekaj lastnosti matrik L_k:

1) L_k je spodnje trikotna
2) $L_k^{-1} = I_n + \ell_k \cdot e_k^T$

D:

$$L_k L_k^{-1} = (I_n - \ell_k e_k^T)(I_n + \ell_k e_k^T) =$$
$$= I_n - \cancel{\ell_k e_k^T} + \cancel{\ell_k e_k^T} - \ell_k \underbrace{(e_k^T \ell_k)}_{} e_k^T =$$
$$= I_n \qquad [0, \dots, 0, 1, 0, \dots, 0] \begin{bmatrix} 0 \\ \vdots \\ x_{k+1}/x_k \\ \vdots \\ * \end{bmatrix} = 0$$

3) imeli bomo $L_n \dots L_1 A = U$

$$A = \underbrace{L_1^{-1} \dots L_{n-1}^{-1}}_{L} U$$

$i < j$

$$L_i^{-1} \cdot L_j^{-1} = (I_n + \ell_i e_i^T)(I_n + \ell_j e_j^T) =$$
$$= I_n + \ell_i e_i^T + \ell_j e_j^T + \ell_i (e_i^T \ell_j) e_i^T =$$
$$= I_n + \ell_i e_i^T + \ell_j e_j^T = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & (\ell_i)_{i+1} & \ddots & & \\ & & & \vdots & & 1 & \\ & & & \vdots & & (\ell_j)_{j+1} & \ddots \\ & & & (\ell_i)_n & & (\ell_j)_n & & 1 \end{bmatrix}$$

$$\qquad \uparrow_i \qquad \uparrow_j$$

iz 3 sledi $L_1^{-1}L_2^{-1}\dots L_{n-1}^{-1} = \begin{bmatrix} 1 & & & & \\ \underset{2:n}{\ell_1} & 1 & & & \\ & \underset{3:n}{\ell_2} & \ddots & & \\ & & \cdots & \ddots & \\ & & & (\ell_{n-1})_n & 1 \end{bmatrix}$

Kako izvedemo LU razcep z elementarnimi eliminacijami?

1. korak: Poiščemo L_1 tako, da v prvem stolpcu A postavimo elemente od 2. do n-tega na nič:

$$L_1 = I - \ell_1 e_1^T; \qquad \ell_1 = \begin{bmatrix} 0 \\ a_{21}/a_{11} \\ \vdots \\ a_{n1}/a_{11} \end{bmatrix}$$

$$L_1 A = (I - \ell_1 e_1^T)A =$$

$$= \left(I - \begin{bmatrix} 0 \\ \ell_{21} \\ \vdots \\ \ell_{n1} \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}\right) A =$$

$$= A - \tilde{A} = \left.\begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ 0 & \\ \vdots & a_{ij}^{(1)} \\ 0 & \end{bmatrix}\right\} A^{(1)} \qquad a_{ij}^{(1)} = a_{ij} - \ell_{i1} a_{1j}$$

2. korak: Poiščemo elementarno eliminacijo L_2 s katero „uničimo" vse elemente v 2. stolpcu matrike $A^{(1)}$ od 3 do n

Primer:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 6 & 4 & 5 \\ 3 & 10 & 8 & 10 \\ 1 & 6 & 6 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2/1 & 2 & 2 & 1 \\ 3/1 & 4 & 5 & 4 \\ 1/1 & 4 & 5 & 7 \end{bmatrix}$$

... zg. trikotnik

... sp. trikotnik

$$\frac{6-2\cdot 2}{1}$$

$$\longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

$5-2\cdot 1$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

LU RAZCEP Z DELNIM PIVOTIRANJEM

$\det(A) \neq 0$, $n \times n$ matrika

Primer:

$$A = \begin{bmatrix} 0 & & \\ * & & \\ \vdots & * & \\ * & & \end{bmatrix} \xrightarrow{\text{menjava}} \tilde{A} = \begin{bmatrix} * & \cdots & * \\ \vdots & & \\ 0 & & \\ \vdots & & \\ * & & \end{bmatrix}$$

$\neq 0$ → iščemo tistega, ki je največji po abs. vrednosti v prvem stolpcu

(da imamo čim manjše numerične napake)

dobimo: $PA = LU$

permutacijska (lahko samo z vektorjem)

Vemo, če je matrika A diagonalno dominantna po stolpcih, ni menjav.

$$|a_{jj}| > \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{kj}|, \quad j = 1, \ldots, n$$

je matrika diagonalno dom. po stolpcih

delno pivotiranje = menjamo samo znotraj stolpca, ne cele matrike

kompletno pivotiranje = menjaš vrstice in stolpce
$PAQ = LU$

↳ se ne uporablja

Primer:

$$A = \begin{bmatrix} 0 & 4 & 12 & 12 \\ 12 & 4 & 8 & 0 \\ 0 & 1 & 9 & 18 \\ 6 & 6 & 8 & 8 \end{bmatrix} \xrightarrow{(1,2)} \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 1 & 9 & 18 \\ 6 & 6 & 8 & 8 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 1 & 9 & 18 \\ 1/2 & 4 & 4 & 8 \end{bmatrix} \xrightarrow{\text{ni piv.}} \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 1/4 & 6 & 15 \\ 1/2 & 1 & -8 & -4 \end{bmatrix} \xrightarrow{(3,4)} \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 1/2 & 1 & -8 & -4 \\ 0 & 1/4 & 6 & 15 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 1/2 & 1 & -8 & -4 \\ 0 & 1/4 & -3/4 & 12 \end{bmatrix}$$

↳ $15 - \left(-\frac{3}{4}\right) \cdot (-4)$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 0 & 1/4 & -3/4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 12 & 4 & 8 & 0 \\ 0 & 4 & 12 & 12 \\ 0 & 0 & -8 & -4 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

MATLAB: $lu(A) = [L, U, P]$

ali samo $lu(A) = [L, U]$

ne $\exists$ možnost, da ne pivotira

21.11.2025

Imam $\quad PA = LU \qquad O(n^3)$

2) $PAx = Pb \quad$ (P obrnljiva)

↳ ohranijo se rešitve

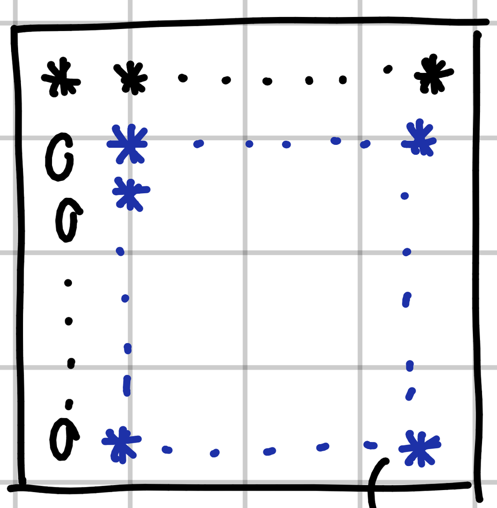
$LUx = Pb = c$

$\to Ly = c$

$\quad Ux = y \qquad O(n^2)$

Algoritem LU razcep (brez pivotiranja)

matrika U:



spremenijo se

```
for j = 1 : n-1
    for i = j+1 : n
        aij = aij / ajj
        for k = j+1 : n
            aik = aik - aij · ajk   #Gaussova eliminacija
        end
    end
end
```

$$\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\Big(\overset{1 \text{ deljenje}}{1} + \sum_{k=j+1}^{n} 2\Big) = \sum_{j=1}^{n-1}\sum_{i=j+1}^{n}\big(1 + 2(n-j)\big) =$$

$$= \sum_{j=1}^{n-1}(n-j)\big(1+2(n-j)\big) = \sum_{j=1}^{n-1}\big(n + 2n^2 - 2nj - j - 2nj + 2j^2\big) =$$

$$= \sum_{j=1}^{n-1}\big(2n^2 + n - 4nj + 2j^2 - j\big) =$$

$$= 2n^2(n-1) + n(n-1) - 4n\sum_{j=1}^{n-1} j + 2\sum_{j=1}^{n-1} j^2 - \sum_{j=1}^{n-1} j =$$

$$= 2n^3 - 2n^2 + n^2 - n - 4n\cdot\frac{(n-1)n}{2} + 2\,\frac{(n-1)n(2n-1)}{6} - \frac{(n-1)n}{2} =$$

$$= \cancel{2n^3} - \cancel{2n^3} + \frac{2}{3}n^3 + O(n^2)$$

$$= \frac{2}{3}O(n^3)$$

če imaš redko (sparse) matriko

$\to$ sparse reprezentacija

tabela $\quad a_{ij} \mid i \mid j \quad$ (tabela $n \times 3$)

$a_{ij} \neq 0$

Uporaba LU razcepa:

1) reševanje večih sistemov z isto matriko

$$A x_i = b_i \quad i = 1, 2, \ldots$$

$\rightarrow$ $PA = LU$ (naredimo 1x)

ponovimo za $i = 1, 2, \ldots$

$$L y_i = P b_i$$
$$U x_i = y_i$$

(če bi Gaussovo elim., bi moral vsakič sproti $A|b$ računati)

reševanje matričnega sistema

$$AX = B \quad , \quad A \in \mathbb{R}^{n \times n} \quad , \quad B \in \mathbb{R}^{n \times p}$$

$$\begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & \cdot \; X \cdot \cdot & x_p \end{bmatrix} = \begin{bmatrix} b_1 & b_2 & \cdot \; B \cdot \cdot & b_p \end{bmatrix}$$

$$A x_1 = b_1$$
$$A x_2 = b_2$$
$$\vdots$$
$$A x_p = b_p$$

časovna zahtevnost:

$$\frac{2}{3} n^3 + O(n^2) + 2p\, O(n^2) =$$
$$= \frac{2}{3} n^3 + O(pn^2)$$

$\rightarrow$ računanje inverza A^{-1}

$$AX = I \Rightarrow X = A^{-1}$$

$$\frac{4}{3} n^3 + O(n^2)$$

2) računanje determinante

$$PA = LU$$
$$\det(PA) = \det(LU)$$
$$\underbrace{\det P}_{(-1)^k} \cdot \det A = \underbrace{\det L}_{1} \cdot \underbrace{\det U}_{\prod_{i=1}^{n} u_{ii}}$$

k ... št. menjav

Primer: $A^{-1} b = ?$ (dana sta A in b)

$$A^{-1} b = x$$
$$b = Ax \Rightarrow \text{reševanje sistema}$$

(bolje kot inverz + sistem)

Analiza napak pri LU razcepu

nisem predelovala za kviz

Rešujemo $Ax=b$, označimo dobljeno rešitev z $\tilde{x}$

1) Koliko se točna rešitev x razlikuje od $\tilde{x}$?

$$\frac{\|x-\tilde{x}\|}{\|x\|} \overset{?}{\leq} \varepsilon$$

2) $\tilde{x}$ je točna rešitev spremenjenega sistema $(A+\delta A)\tilde{x} = (b+\delta b)$
$\rightsquigarrow$ obratna stabilnost, če sta δA in δb majhna

Analiza obratne stabilnosti

Oznaka:
$$|A| = \begin{bmatrix} |a_{11}| & |a_{12}| & \dots & |a_{1n}| \\ \vdots & & & \vdots \\ |a_{n1}| & \dots & \dots & |a_{nn}| \end{bmatrix}$$

za reševanje LU razcepa velja naslednja lema:

Lema: Naj bo $A \in \mathbb{R}^{n\times n}$, ki je ustrezno permutirana. Izvedemo LU razcep in dobimo L in U.
Za izračunani matriki L in U velja:

$$A = LU + E \quad , \quad |E| \leq n \cdot \underset{\substack{\uparrow \\ \text{osnovna zaokrožitvena napaka}}}{u} \cdot |L| \cdot |U|$$

Opomba: za ∞, 1 in Frobeniusovo normo je
$\|A\| = \| \, |A| \, \|$

Lema: Sistem $Ly=b$ rešujemo z direktnim vstavljanjem.
Končna rešitev $\tilde{y}$ zadošča zvezi:

$$(L+\delta L)\tilde{y} = b \quad ; \quad |\delta L| \leq n \cdot u \cdot |L|$$

Lema: Podobno, če rešujemo sistem $Ux=\tilde{y}$, izračunana rešitev $\tilde{x}$ zadošča zvezi

$$(U+\delta U)\tilde{x} = \tilde{y} \quad , \quad |\delta U| \leq n \cdot u \cdot |U|$$

$$\Rightarrow \quad b = (L+\delta L)\tilde{y} = (L+\delta L)(U+\delta U)\tilde{x} =$$

$$= (LU + L\delta U + \delta L U + \delta L \delta U)\tilde{x} =$$

$$= (A - \underbrace{E + \delta L U + L \delta U + \delta L \delta U}_{\delta A})\tilde{x} =$$

Torej je $(A+\delta A)\tilde{x} = b$

Ocenimo:

$$|\delta A| \leq |E| + |\delta L|\cdot|U| + |L|\cdot|\delta U| + |\delta L|\cdot|\delta U| \leq$$

$$\leq n\cdot u\cdot |L|\cdot|U| + n\cdot u\cdot |L|\cdot|U| + n\cdot u |L|\cdot|U| + n^2u^2 |L|\cdot|U|$$

$$\approx 3nu |L|\cdot|U|$$

$$\Rightarrow \| |\delta A| \|_\infty = \|\delta A\|_\infty \leq 3nu \|L\|_\infty \cdot \|U\|_\infty$$

$\|L\|_\infty \leq n$

$|l_{ij}| \leq 1$, ker je kvocient nekega elementa z največjim v istem stolpcu

$$\|L\|_\infty \leq \sum_{i=1}^{n} 1 = n$$

Definiramo:

$$g = \frac{\max_{i,j} |u_{ij}|}{\max_{i,j} |a_{ij}|} \quad \ldots \quad \text{pivotna rast}$$

(pove koliko se elementi v U večajo)

$$\|U\|_\infty \leq n\cdot \max_{i,j} |u_{ij}| = n\cdot g\cdot \max_{i,j} |a_{ij}| \leq n\cdot g\cdot \|A\|_\infty$$

$$\|\delta A\|_\infty \leq 3\cdot nu\cdot n\cdot n\cdot g\cdot \|A\|_\infty$$

$$\frac{\|\delta A\|_\infty}{\|A\|_\infty} \leq 3u\cdot n^3 g$$

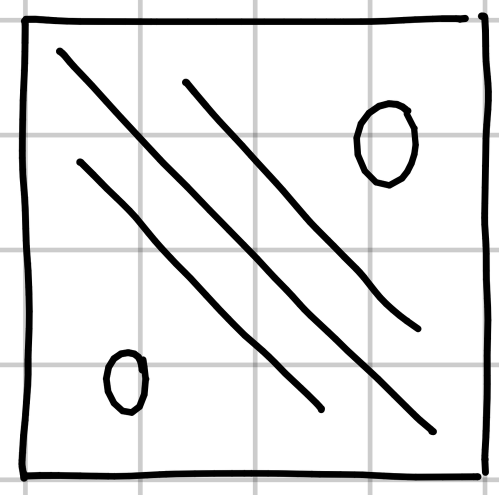
Vemo, da je pri delnem pivotiranju pivotna rast omejena $g \leq 2^{n-1}$

Običajno je $g \approx n^{2/3}$

(pri kompletnem je v povprečju $g \approx n^{1/2}$, kar ni precej boljše)

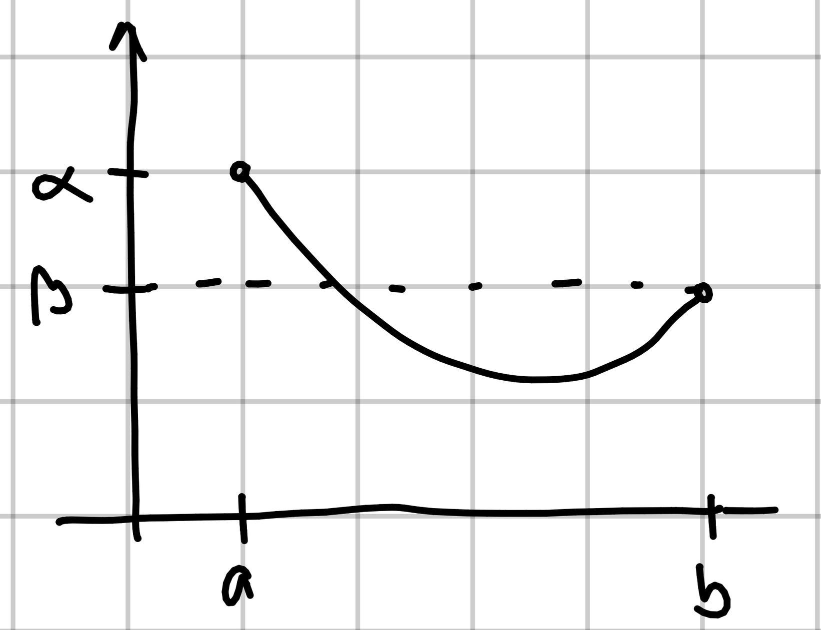
SISTEMI S POSEBNO OBLIKO

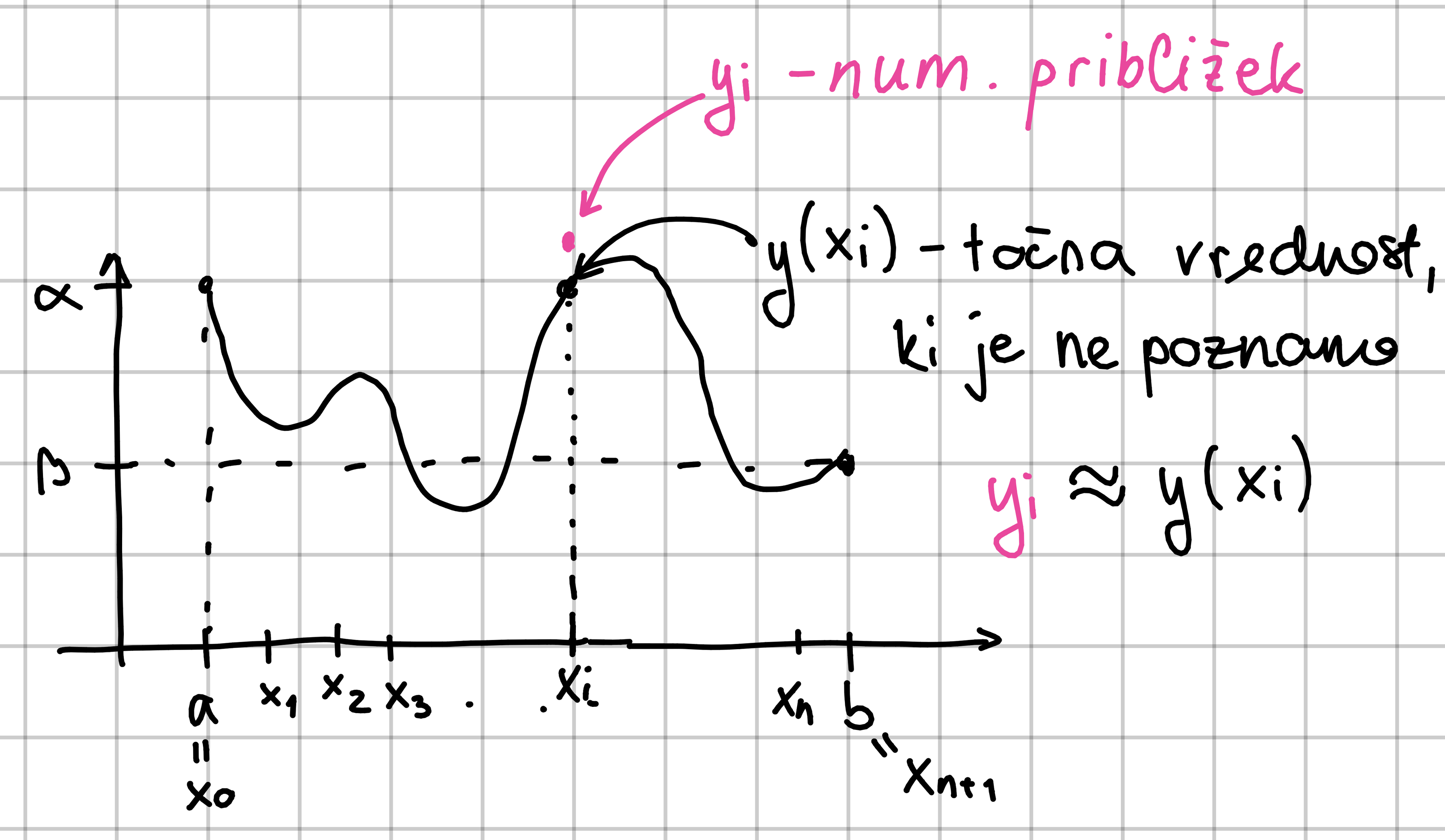
- tridiagonalna matrika



- pasovne matrike (širine k)
  ↳ tripasovne so širine 1
- simetrične pozitivno definitne matrike
- kompleksne matrike
  (razcepimo na realni in kompleksni del)

Metoda končnih diferenc





$x_{i+1} - x_i =: h$ $\qquad n >> 1$

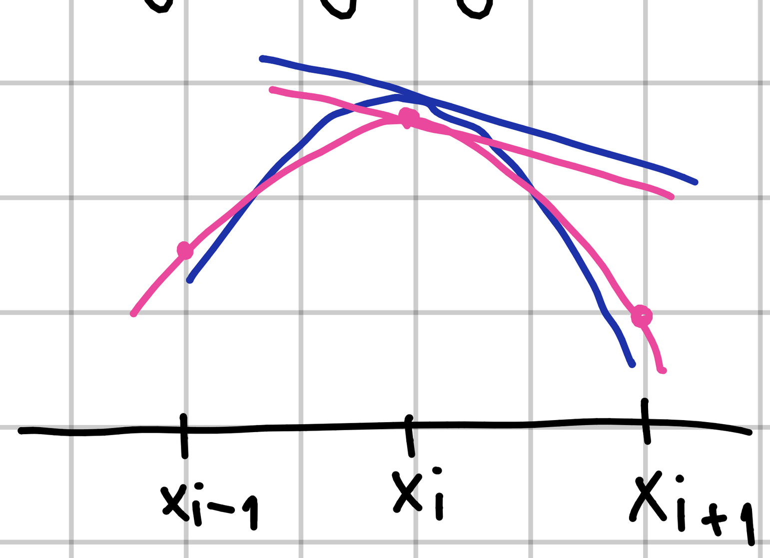
Recimo, da zahtevamo „samo"

$$-y''(x_i) + \underbrace{p(x_i)}_{p_i}y'(x_i) + \underbrace{q(x_i)y(x_i)}_{q_i \cdot y_i} = \underbrace{r(x_i)}_{r_i} \qquad i = 0, \underbrace{1, \dots, n}_{\text{neznanke}}, n+1$$

neznanke: y_i, $i = 1, 2, \dots, n$

$\downarrow$

y_{i-1}, y_i, y_{i+1}



potegneš parabolo skozi te 3 točke in tangento na to parabolo

$$A = \begin{bmatrix} * & * & 0 & \cdots & \cdots & 0 \\ * & * & * & 0 & \cdots & 0 \\ 0 & * & * & * & 0 \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & * & * & * \\ 0 & \cdots & & 0 & * & * \end{bmatrix}$$

tridiagonalni sistem

↳ lahko reši v $O(n)$
algoritem na vajah

Simetrične pozitivno definitne matrike (s.p.d.)

DEF.: Matrika $A \in \mathbb{R}^{n\times n}$ je s.p.d., če je $A = A^T$ in za vsak $x \neq 0$: $x^T A x > 0$.

Lastnosti:

- Naj bo C obrnljiva, potem je A s.p.d. $\Leftrightarrow$ $C^T A C$ s.p.d.
- Vse vodilne podmatrike s.p.d. matrike A so s.p.d.
- Naj bo A s.p.d., potem je tudi matrika $H = A([i_1, i_2, \ldots, i_k], [i_1, i_2, \ldots, i_k])$ s.p.d.
- Če je A s.p.d., so vse lastne vrednosti pozitivne.
  Dokaz:
  (vse l. vrednosti simetričnih matrik so realne, l. vektorji so ortogonalni)
  (λ, v) ... lastni par $(v \neq 0)$
  $v^T A v > 0$
  $v^T(Av) = v^T(\lambda v) = \lambda v^T v = \lambda \|v\|_2^2 > 0$
  $\Rightarrow \lambda > 0$
- Če je A s.p.d., potem $\exists$ LU razcep brez pivotiranja in $u_{ii} > 0$.
  $\rightsquigarrow$ iz 2. lastnosti $\exists$ LU

IZREK: Matrika A je s.p.d. $\Leftrightarrow$ $\exists$ taka (nesingularna) spodnja trikotna matrika V s pozitivnimi diagonalnimi elementi, da je $A = VV^T$ (razcep Choleskega).

Dokaz:

$(\Leftarrow)$: $A = VV^T \Rightarrow A^T = (VV^T)^T = (V^T)^T V^T = VV^T = A$
$\Rightarrow A$ je simetrična

$x \neq 0$: $x^T A x > 0$

$$x^T A x = x^T (VV^T) x = (x^T V)(V^T x) = (V^T x)^T (V^T x) =$$
$$= \|V^T x\|_2^2 \geq 0$$

Če bi bilo $\|V^T x\|_2 = 0$, potem (def.) je $V^T x = 0$
V^T je nesingularna, zato ne more preslikati $x \neq 0$ v 0.

$(\Rightarrow)$: Vemo, da $\exists$ LU razcep brez pivotiranja $(u_{ii} > 0)$.

$A = LU$;

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

$$A = L \cdot D \cdot M$$

$$D = \begin{bmatrix} u_{11} & & & \\ & u_{22} & & \\ & & \ddots & \\ & & & u_{nn} \end{bmatrix} \qquad M = \begin{bmatrix} \frac{u_{11}}{u_{11}} & \frac{u_{12}}{u_{11}} & \dots & \frac{u_{1n}}{u_{11}} \\ 0 & \frac{u_{22}}{u_{22}} & & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \frac{u_{nn}}{u_{nn}} \end{bmatrix}$$

Ker je $A = A^T$ je

$$A^T = LDM = M^T(D^T L^T) = \underbrace{M^T}_{\text{spodnje trikotna z 1 na diagonali}} (\overbrace{D L^T}^{\text{zgornje trikotna}})$$ je LU razcep

matrike A (A^T). Ker je enoličen, mora biti $L = M^T$.
Torej je $A = LDL^T$ in zato je $V = L \cdot D^{1/2}$

↳ to je ok, ker so diagonalci pozitivni

V je spodnje trikotna s pozitivnimi elementi na diagonali. □

Algoritem:

$$A = VV^T; \quad V = \begin{bmatrix} v_{11} & 0 & \dots & 0 \\ v_{21} & v_{22} & \ddots & \\ \vdots & \vdots & & \ddots \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix}$$

$$\boxed{\begin{array}{c} \underbrace{e_i^T A e_i}_{a_{ii}} > 0 \\ a_{ii} > 0 \end{array}}$$

$$VV^T = \begin{bmatrix} v_{11} & 0 & \dots & 0 \\ v_{21} & v_{22} & & \\ \vdots & \vdots & \ddots & \\ v_{n1} & v_{n2} & \dots & v_{nn} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} & \dots & v_{n1} \\ 0 & v_{22} & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & v_{nn} \end{bmatrix}$$

- Računamo stolpce V zaporedoma

1. stolpec:

$$V \cdot \begin{bmatrix} v_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v_{11} \cdot \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$$

$$v_{11}^2 = a_{11}$$
$$v_{11} = \oplus\sqrt{a_{11}}$$

↳ hočemo poz. diagonalce

$$v_{11} \cdot v_{j1} = a_{j1} \qquad j = 2, \dots, n$$
$$v_{j1} = a_{j1} / v_{11}$$

stolpci od 2 do n:

Recimo, da smo že izračunali prvih j-1 stolpcev matrike V.

j-ti stolpec:

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = V \cdot \begin{bmatrix} v_{1j} \\ v_{2j} \\ \vdots \\ v_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = v_{j1} \cdot \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{n1} \end{bmatrix} + v_{j2} \cdot \begin{bmatrix} 0 \\ v_{22} \\ \vdots \\ v_{n2} \end{bmatrix} + \dots + v_{jj} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_{jj} \\ \vdots \\ v_{nj} \end{bmatrix}$$

$$a_{jj} = v_{j1}^2 + v_{j2}^2 + \dots + v_{jj}^2$$

$$\rightsquigarrow \quad v_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} v_{jk}^2}$$

$$i > j: \quad a_{ij} = v_{i1} v_{j1} + v_{i2} v_{j2} + \dots + v_{ij} v_{jj}$$

$$\rightsquigarrow \quad v_{ij} = \frac{1}{v_{jj}} \cdot \left(a_{ij} - \sum_{k=1}^{j-1} v_{ik} v_{jk} \right)$$

for $j = 1 : n$

  $v_{jj} = \sqrt{a_{jj} - \sum_{k=1}^{j-1} v_{jk}^2}$

  for $i = j+1 : n$

    $v_{ij} = 1/v_{jj} \left(a_{ij} - \sum_{k=1}^{j-1} v_{ik} v_{jk} \right)$

  end

end

št. operacij: $\frac{1}{3} n^3 + O(n^2)$

Razcep Choleskega lahko uporabimo za test, ali je A s.p.d.

Pivotiranje?

→ Diagonalno pivotiranje: na vsakem koraku poiščemo največji element (po abs. vrednosti) na preostanku diagonale.

MATLAB:

naredim random matriko A

→ $B = A^T A$ je s.p.d.

  $V = \text{chol}(B)$

če A ni p.d., to javi $\rightsquigarrow$ test za to

SISTEMI NELINEARNIH ENAČB

$$\left.\begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0 \\ f_2(x_1, \dots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \dots, x_n) &= 0 \end{aligned}\right\}$$ Vsaj 1 funkcija f_i je nelinearna.

$$\left.\begin{aligned} F &= [f_1, \dots, f_n]^T \\ X &= [x_1, x_2, \dots, x_n]^T \end{aligned}\right\} \Rightarrow F(X) = 0$$

Navadna iteracija:

$$F(X) = 0 \iff X = G(X)$$

Tvorimo $X^{(k+1)} = G(X^{(k)})$; $X^{(0)}$ izberemo

$$\Omega \subseteq \mathbb{R}^n: \quad G: \Omega \to \Omega$$

IZREK: Naj bo $G: \Omega \to \Omega$ preslikava, za katero je:

1) $X \in \Omega \Rightarrow G(X) \in \Omega$
2) G je skrčitev na Ω: $X, Y \in \Omega$: $\|G(X) - G(Y)\| \leq q\|X - Y\|$
$0 < q < 1$

Potem zaporedje $(X^{(k)})_{k=1}^{\infty}$: $X^{(k+1)} = G(X^{(k)})$ konvergira proti negibni točki $\alpha \in \Omega$: $\alpha = G(\alpha)$, ki je rešitev sistema $F(X) = 0$, $\forall X^{(0)} \in \Omega$!

POSLEDICA: G je skrčitev, če je $\rho(J_G(X)) < 1 \quad \forall X \in \Omega$

$\uparrow$ spektralni radij

$$J_G(X) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(X) & \frac{\partial g_1}{\partial x_2}(X) & \dots & \frac{\partial g_1}{\partial x_n}(X) \\ \vdots & & & \\ \frac{\partial g_n}{\partial x_1}(X) & \dots & & \frac{\partial g_n}{\partial x_n}(X) \end{bmatrix}$$

$$G = [g_1 \dots g_n]^T$$

(Newton-Raphson method)

Posplošitev Newtonove metode:

$$x_{n+1} = \underbrace{x_n - \frac{f(x_n)}{f'(x_n)}}_{g(x_n)}$$

$$G(X) = X - J_F^{-1}(X) \cdot F(X)$$

$$X^{(k+1)} = X^{(k)} - J_F^{-1}(X^{(k)}) \cdot F(X^{(k)}) \; ; \; k = 0, 1, 2, \ldots$$

$\frac{a}{b} = a \cdot b^{-1} = b^{-1} \cdot a$

→ ne moreš za matrike

$X^{(0)}$ izberemo

$$X^{(k+1)} - X^{(k)} = -J_F^{-1}(X^{(k)}) \cdot F(X^{(k)}) \quad / \cdot J_F(X^{(k)}) \text{ z leve}$$

$$J_F(X^{(k)})\underbrace{(X^{(k+1)} - X^{(k)})} = -F(X^{(k)})$$

$$\underset{A}{J_F(X^{(k)})} \cdot \underset{x}{\Delta X^{(k)}} = \underset{b}{-F(X^{(k)})} \Rightarrow X^{(k+1)} = X^{(k)} + \Delta X^{(k)}$$

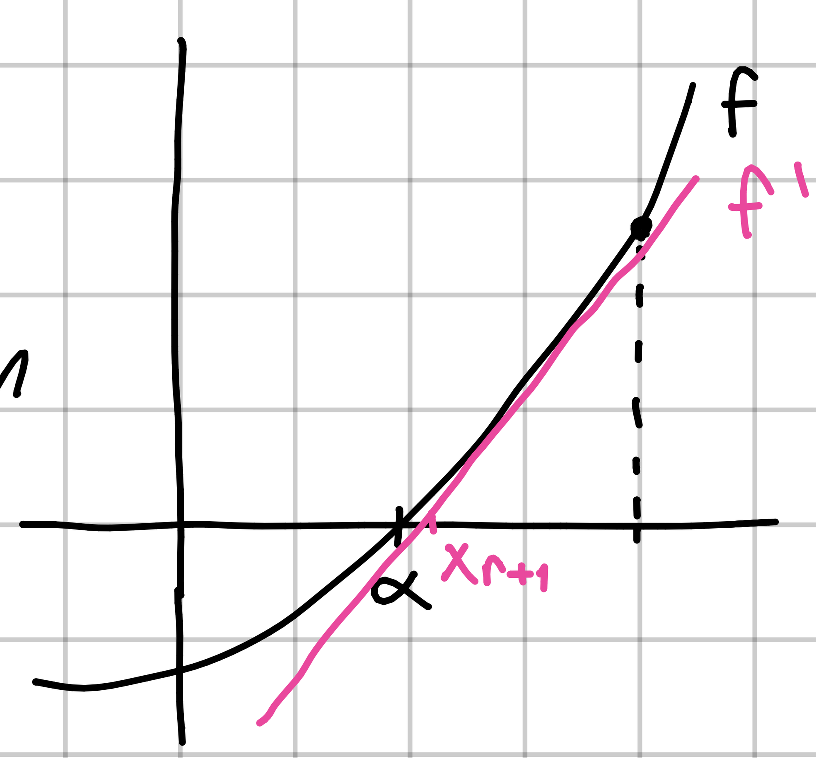
$$\| \Delta X^{(k)} \| < \varepsilon$$

Če je sistem linearen, se po 1 koraku ustavi

Izrek Kantoroviča zagotavlja konvergenco ob (relativno zapletenih) pogojih.
Ponavadi je konvergenca kvadratična.

Kvazi - Newtonova metoda

Ne računamo $J_F(X^{(k)})$ na vsakem koraku, ampak samo na vsakih nekaj korakov.



Začetni približek?

Primer:

Reševanje $\underbrace{z^3 + 1 = 0}$, $z \in \mathbb{C}$ ⟶ rešitve: $z_1 = -1$, $z_2 = \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}$, $z_3 = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$

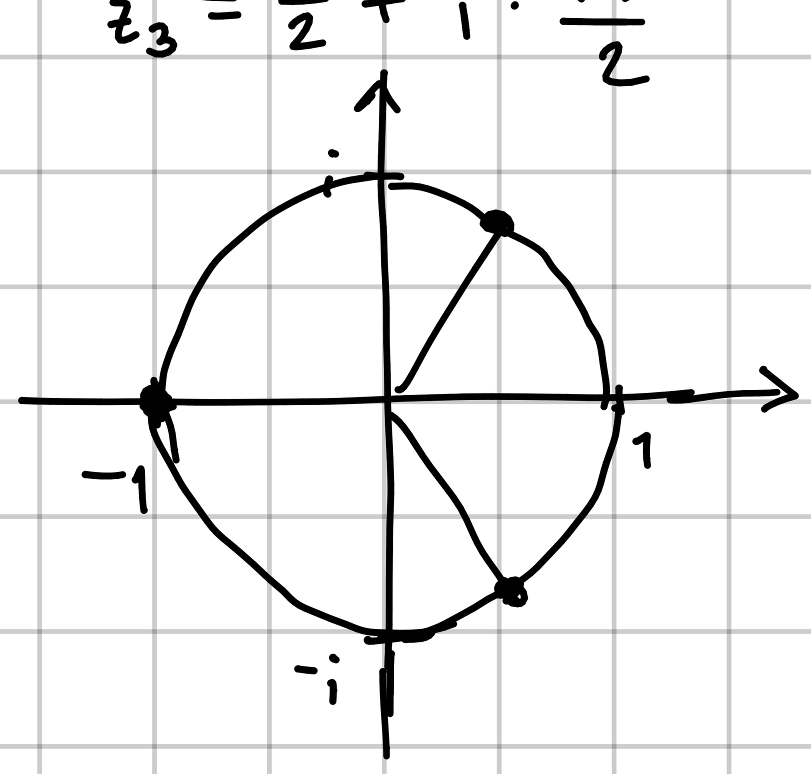
$z = x + iy \; ; \; x, y \in \mathbb{R}$

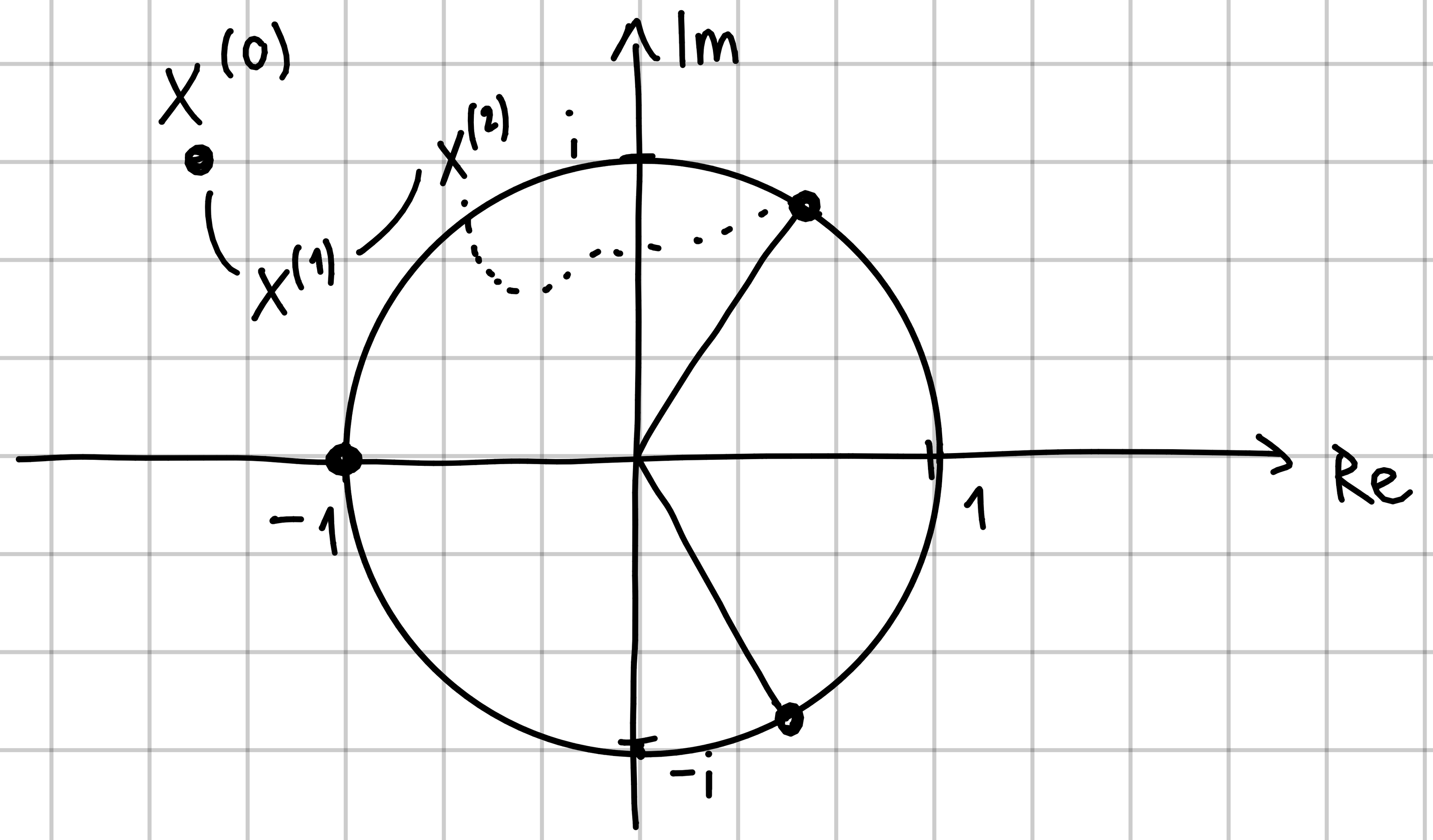
$$z^3 + 1 = x^3 + 3ix^2y - 3xy^2 - iy^3 + 1 = 0$$

$$\mathrm{Re}(z^3 + 1) = x^3 - 3xy^2 + 1 = 0$$
$$\mathrm{Im}(z^3 + 1) = 3x^2y - y^3 = 0$$

↓

rešitve: $(-1, 0)$, $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$





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$$\left.\begin{aligned} f_1(x_1, \dots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, \dots, x_n) &= 0 \end{aligned}\right\} \quad \begin{aligned} F &= (f_1, f_2, \dots, f_n)^T \\ X &= (x_1, \dots, x_n)^T \end{aligned}$$

$$\Updownarrow$$

$(f(x)=0)$

$$F(X) = 0 \iff X = G(X)$$

$$X^{(r+1)} = G(X^{(r)}), \ X^{(0)} \text{ izberemo}$$

$$F(\alpha) = 0 \iff \alpha = G(\alpha)$$

Izpeljava Newtonove metode:

$F(X) = 0$

imamo $X^{(r)}$, ki je približek

Iščemo „popravek" $\Delta X^{(r)}$: $F(X^{(r)} + \Delta X^{(r)}) = 0$

$$F(x) = \begin{bmatrix} f_1(X) \\ \vdots \\ f_n(X) \end{bmatrix}$$

$X = (x_1, \dots, x_n)^T$, $\Delta X = (\Delta x_1, \dots, \Delta x_n)^T$

$f_i : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$

$$f_i(X + \Delta X) = f_i(X) + \frac{\partial f_i}{\partial x_1} \Delta x_1 + \dots + \frac{\partial f_i}{\partial x_n} \Delta x_n + \dots$$

→ od tu naprej ne vzamemo, ker potem ne bi imeli lin. sistema

$$F(X + \Delta X) = 0 \approx F(X) + \underbrace{\begin{bmatrix} \nabla f_1(X) \\ \vdots \\ \nabla f_n(X) \end{bmatrix}}_{J_F(X)} \cdot \Delta X$$

Rešujemo: $0 = F(X) + J_F(X) \cdot \Delta X$

$$J_F(X) \, \Delta X = -F(X)$$

Kako se izogniti računanju odvodov?

→ Namesto $J_F(X^{(r)})$ vzamemo približek za to matriko

Naj bo B_r približek za $J_F(X^{(r)})$

a) rešimo sistem $B_r \Delta X^{(r)} = -F(X^{(r)})$

b) $X^{(r+1)} = X^{(r)} + \Delta X^{(r)}$

c) določimo B_{r+1} za naslednji korak

Broydenova metoda

za B_{r+1} vzamemo v spektralni normi najbližjo matriko B_r, ki zadošča „sekantnemu pogoju":

$$B_{r+1}(X^{(r+1)} - X^{(r)}) = F(X^{(r+1)}) - F(X^{(r)})$$

Rešitev (brez dokaza): $$B_{r+1} = B_r + \frac{F(X^{(r+1)}) \cdot \Delta X^{(r)T}}{\Delta X^{(r)T} \cdot \Delta X^{(r)}}$$

- Na začetku za B_0 izberemo čim boljšo aproksimacijo $J_F(X^{(0)})$, pogosto kar I.

Variacijske metode

$f_1(X)=0, \ldots, f_n(X)=0$

$\Updownarrow$

$G(X) = f_1^2(X) + \cdots + f_n^2(X)$ ima globalni minimum enak 0

Iščemo minimum funkcije
$G: \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^n$
$(G(X) = \sum f_i^2(X))$

Potreben pogoj za min: $\nabla G(X) = 0$ (ni zadosten, npr. $-x^2$)

Kdaj je točka res minimum?

$$H_G(X) = \begin{bmatrix} \frac{\partial^2 G}{\partial x_1^2}(X) & \cdots & \frac{\partial^2 G}{\partial x_1 \partial x_n}(X) \\ \vdots & & \\ \frac{\partial^2 G}{\partial x_n \partial x_1}(X) & \cdots & \frac{\partial^2 G}{\partial x_n^2}(X) \end{bmatrix}$$

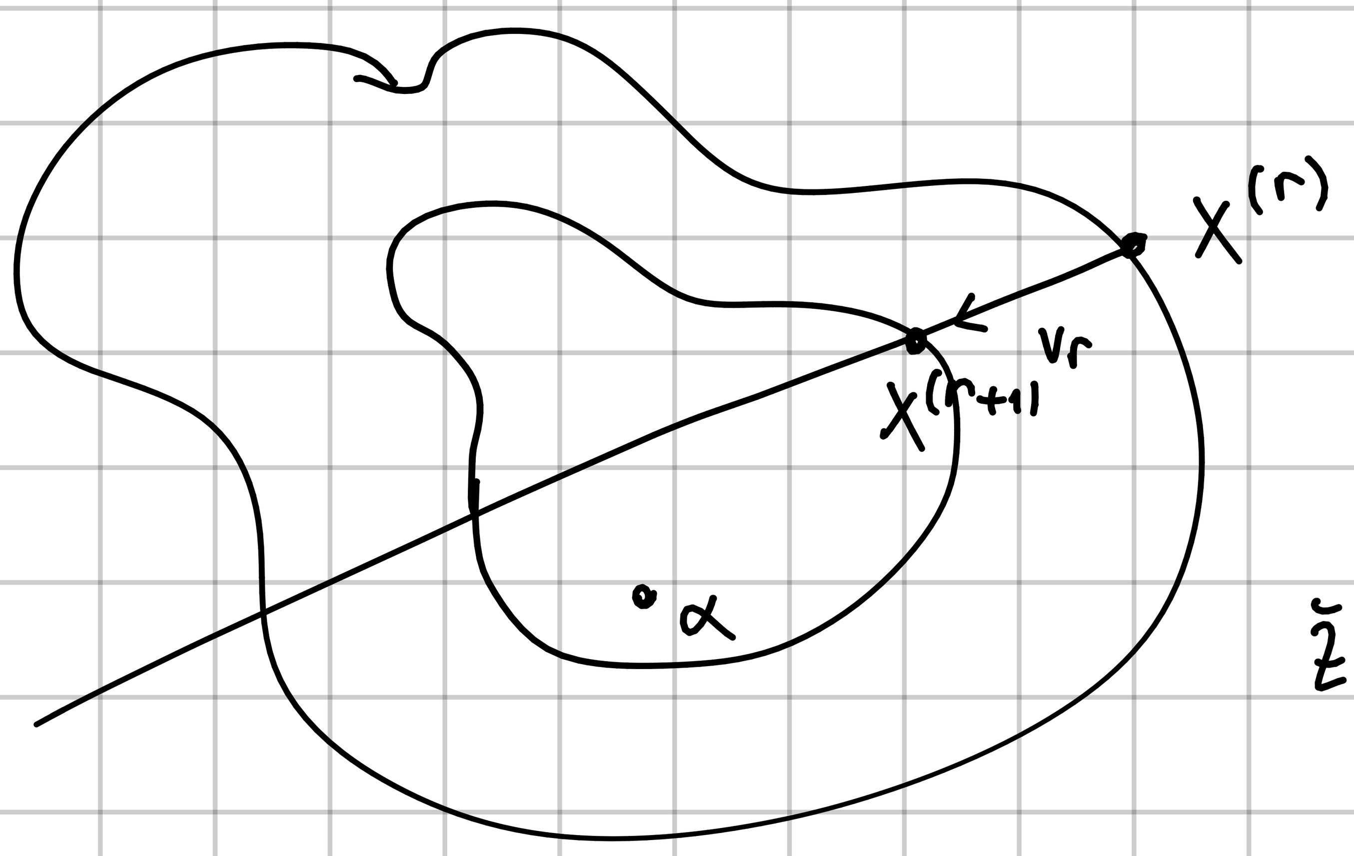
Hessejeva matrika

1) Če je $H_G(X)$ pozitivno definitna, je pri X minimum.
2) Če je $H_G(X)$ negativno definitna, je pri X maksimum.
3) Če je $H_G(X)$ nedefinitna, potem ne moremo sklepati na min ali max.

Za G vzamemo: $G(X) = \sum_{i=1}^{n} f_i^2(X)$

Splošni pristop:

- Naj bo $x^{(r)}$ trenutni približek
  → izberemo smer $v_r \in \mathbb{R}^n$ in $\lambda_r \in \mathbb{R}$ tako, da bo vrednost G v novem približku $x^{(r+1)} = x^{(r)} + \lambda_r v_r$ manjša



$$x^{(r+1)} = x^{(r)} + \lambda_r v_r$$

želimo: $G(x^{(r+1)}) < G(x^{(r)})$

Kako izbrati v_r in λ_r?

a) splošna metoda spusta:
izberemo poljubno smer v_r, ki ni pravokotna na $\nabla G(x^{(r)})$ (da ne bomo ostali na izohipsi)

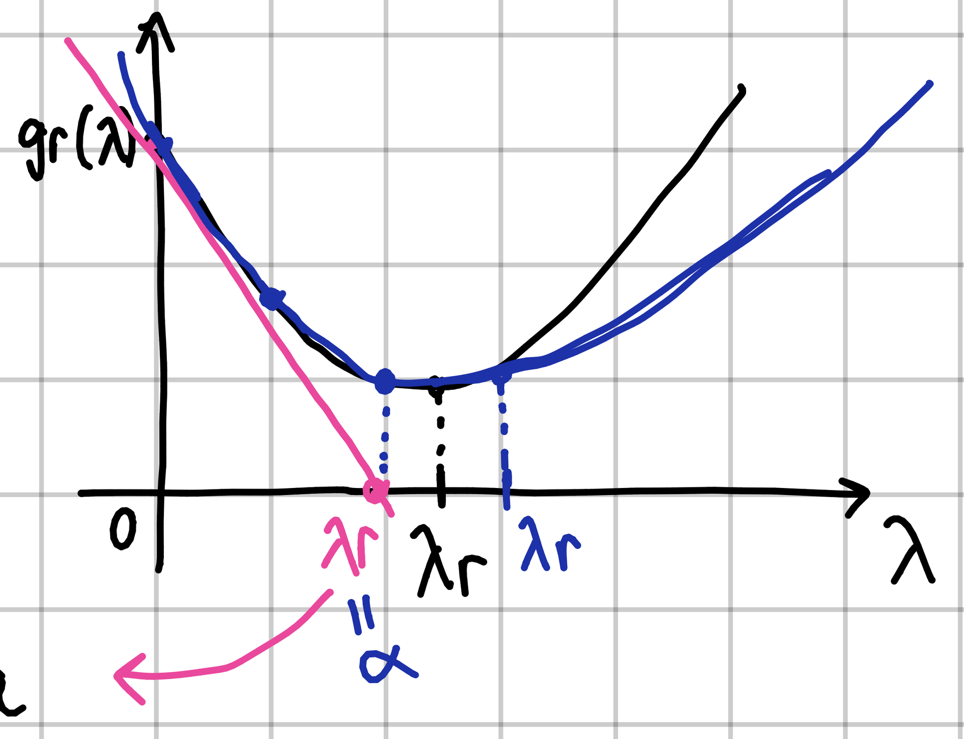
b) metoda najhitrejšega spusta:
$v_r = -\nabla G(x^{(r)})$

c) metoda koordinatnega spusta:
za smeri ciklično izbiramo t.i. koordinatne smeri $e_1, e_2, \dots, e_n$

Kako določamo λ_r?
→ pri določanju gledamo funkcijo
$$g_r(\lambda) = G(x^{(r)} + \lambda v_r)$$
ene spremenljivke in določimo λ_r tako, da je
$$g_r(\lambda_r) < g_r(0)$$
variante:

a) metoda največjega spusta
λ_r določimo tako, da poiščemo minimum funkcije g_r: $g'(\lambda) = 0$

Kako izračunamo $g'(\lambda)$?



b) metoda tangentnega spusta
za λ_r vzamemo presek tangentne na g_r pri $\lambda = 0$ z „x-osjo"

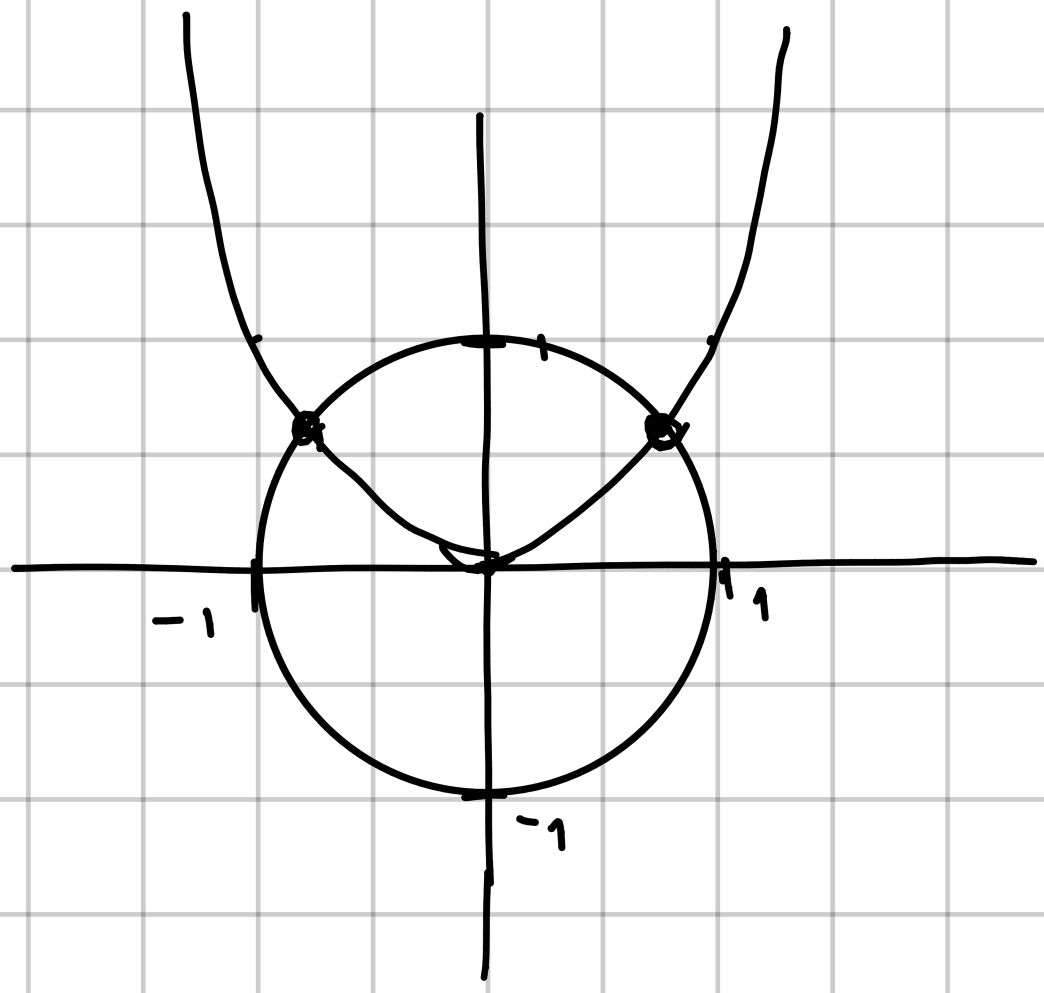
c) metoda paraboličnega spusta *

s tangentno metodo določimo $\alpha = \lambda r$, nato pa skozi točke $(0, g_r(0))$, $(\alpha/2, g_r(\frac{\alpha}{2}))$ in $(\alpha, g_r(\alpha))$ potegnemo parabolo.

Nato za λr vzamemo tisto vrednost, pri kateri parabola doseže minimum.

MATLAB:

$$x^2 + y^2 - 1 = 0$$
$$y - x^2 = 0$$



$$y = x^2 \Rightarrow x^2 + x^4 - 1 = 0$$

$z = x^2$:

$$z^2 + z - 1 = 0$$
$$z_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$$

$\left.\begin{array}{l}\text{fsolve}\\ \text{fminsearch}\end{array}\right\}$ optimization toolbox → treba posebaj naložiti

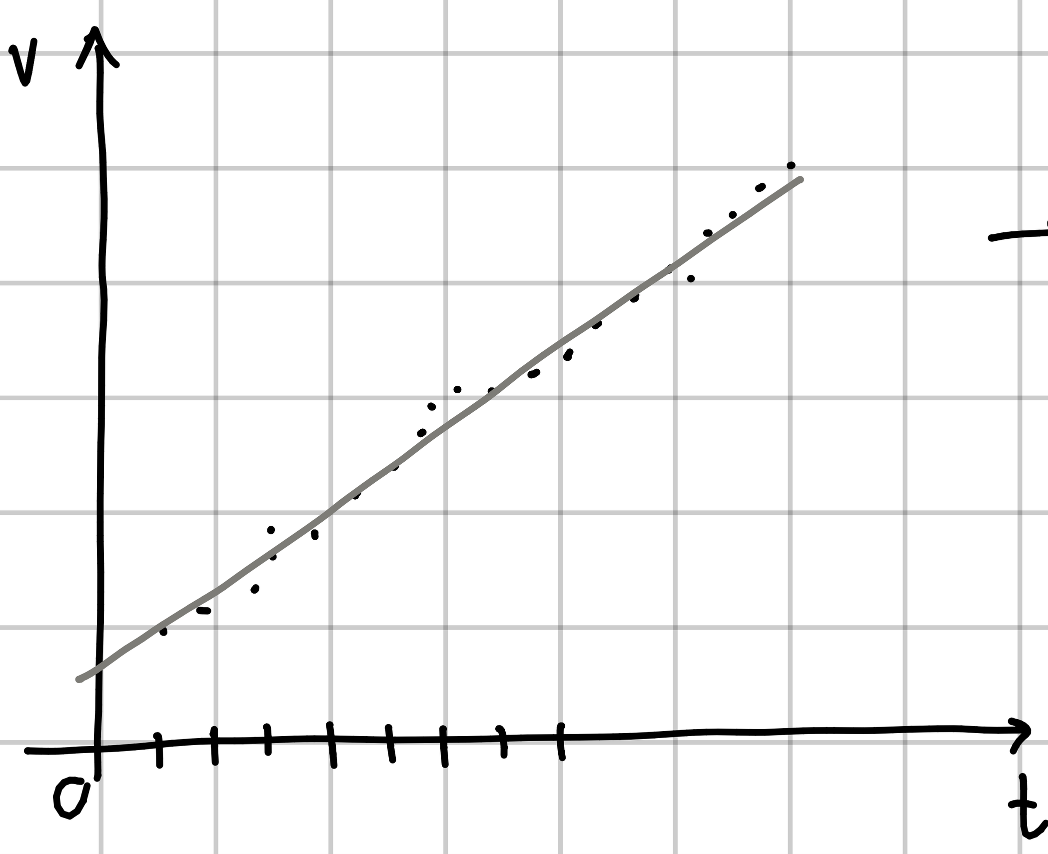
LINEARNI PROBLEM NAJMANJŠIH KVADRATOV

Primer: Recimo, da merimo hitrost objekta ob različnih časih t_i, $i \in [N]$. Denimo, da vemo, da je hitrost oblike
$$v(t) = \alpha t + \beta$$
Kako določimo α, β?

Meritev je zelo veliko $N \gg 1$.
V resnici imamo naslednji problem:
$$\begin{aligned} \alpha t_1 + \beta &= v_1 \\ \alpha t_2 + \beta &= v_2 \\ &\vdots \\ \alpha t_N + \beta &= v_N \end{aligned}$$

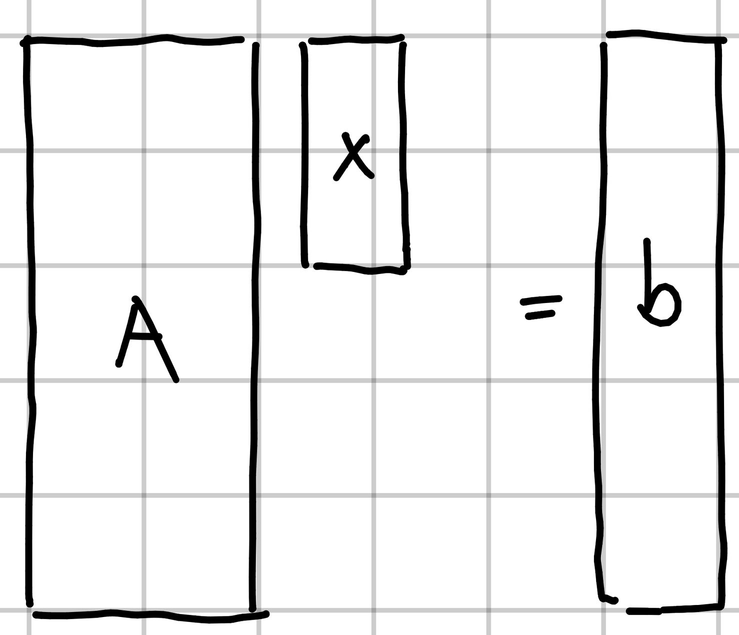
Problem skoraj gotovo nima točne rešitve.
Primer:
$$\begin{aligned} \alpha + \beta &= 1 \\ 2\alpha + \beta &= 3 \\ 3\alpha + \beta &= 5 \end{aligned} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$t_1 \quad 1 \quad \sin(2\pi t_1) \quad \cos(2\pi t_1)$



→ iščemo tako premico

Rešujemo: $Ax = b$, $A \in \mathbb{R}^{m \times n}$; $m \gg n$
$x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$



$x^* = \operatorname{argmin} \|Ax - b\|_2$; x^* je „rešitev" po metodi najmanjših kvadratov

najde input value (argument), ki da najmanjši output, ne pa najmanjše vrednosti tega

Vprašanja: Ali x^* $\exists$? Ali je enoličen? Kako ga dobimo?

Če A ni polnega ranga ($\operatorname{rang} A < n$), potem rešitev (če $\exists$) ni enolična.

rang → največje št. lin. neodvisnih vrstic ali stolpcev v matriki

$$\operatorname{rang}(A) < n \Rightarrow \dim \ker A > 0$$
$$\exists\, z \in \ker A,\ z \neq 0 : Az = 0$$

Recimo, da je x^* optimalna „rešitev" (po metodi naj. kvadratov), potem je rešitev tudi $\tilde{x} = x^* + \lambda z$; $\lambda \in \mathbb{R}$

$$A\tilde{x} - b = \underbrace{A(x^* + \lambda z) - b}_{Ax^* + \underbrace{\lambda Az}_{0} - b} = Ax^* - b$$

Zahtevamo: A ima poln rang ($\operatorname{rang} A = n$)

(v praksi nepoln rang pomeni odvisnost parametrov)

Možni načini reševanja

1) Normalni sistem enačb : $A^T A x = Ab$

→ zakaj $\exists$ natanko 1 rešitev?

$A^T A \in \mathbb{R}^{n \times n}$

Kdaj ima kvadratni sistem natanko 1 rešitev?

→ ko je $A^T A$ obrnljiva

Lema: $A^T A$ je s.p.d.

Dokaz: $(A^T A)^T = A^T A$

$x \neq 0$: $x^T (A^T A) x > 0$

$$x^T (A^T A) x = (x^T A^T)(Ax) = (Ax)^T (Ax) = \|Ax\|_2^2 \geq 0$$

Če je $\|Ax\|_2 = 0$, potem je $Ax = 0$.

$\Rightarrow x \in \ker A \underset{\operatorname{rang}=n}{\Rightarrow} x = 0$ $\nrightarrow$

$\square$

Vse l. vrednosti $A^T A$ so torej pozitivne, zato je $\det A^T A > 0$.

$\Rightarrow A^T A$ obrnljiva

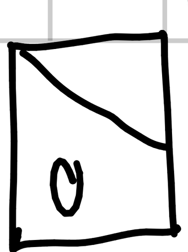
2) QR razcep A: $A = QR$

2.1) Standardni QR razcep : $A = QR$, $Q \in \mathbb{R}^{m \times n}$ ortonormiranimi stolpci

$R \in \mathbb{R}^{n \times n}$ pa zgornje trikotna matrika s pozitivnimi diagonali

2.2) Razširjeni QR razcep : $A = QR$, $Q \in \mathbb{R}^{m \times m}$ je ortogonalna,

$R \in \mathbb{R}^{m \times n}$ je kvazi zgornje trikotna

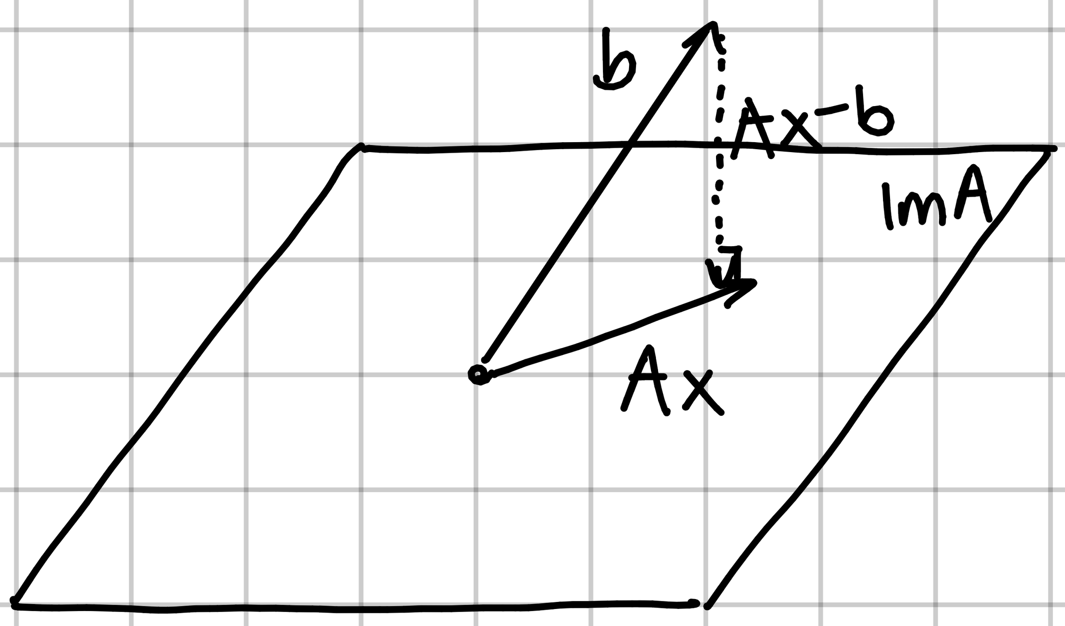


3) Singularni razcep: $A = U \Sigma V^T$; $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n \times n}$ ortogonalni, Σ je kvazi diagonalna

Geometrijska interpretacija normalnega sistema

$A: \mathbb{R}^n \to \mathbb{R}^m$

$\dim \operatorname{Im} A = n$
(ker ima A poln rang)



$x^* = \operatorname{argmin} \|Ax-b\|_2$, če je $b \in \operatorname{Im}$ $\Rightarrow$ je točna rešitev

zaradi 2. norme

$$Ax-b \perp \operatorname{im} A$$
$$\Updownarrow$$
$$Ax-b \perp \text{baza } \operatorname{Im} A = \{a_1, a_2, \dots, a_n\}$$

$$a_i^T(Ax-b) = 0 \quad i = 1, 2, \dots, n$$

$$\begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} (Ax-b) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A^T(Ax-b) = 0$$
$$A^TAx = A^Tb$$

Lema: Vektor $x^* \in \mathbb{R}^n$ je rešitev sistema $A^TAx = A^Tb \iff Ax^*-b$ je pravokoten na $\operatorname{Im} A$.

Dokaz:

($\Rightarrow$): $A^TAx^* = A^Tb$; izberemo $y \in \operatorname{Im} A$: $Ax^*-b \perp y$

$$y^T(Ax^*-b) = (Ax)^T(Ax^*-b) = (x^TA^T)(Ax^*-b) =$$
$$= x^T(A^TAx^* - A^Tb) = x^T \cdot 0 = 0$$

($\Leftarrow$): $Ax^*-b \perp \operatorname{Im} A$: $\forall y \in \operatorname{Im} A$: $y^T(Ax^*-b) = 0$;

$$(A\cdot e_i)^T(Ax^*-b) = 0$$
$$e_i^TA^T(Ax^*-b) = 0$$
$$e_i^T(A^TAx^* - A^Tb) = 0 \qquad i = 1, 2, \dots, n$$

Lema: Vektor x^* je rešitev po metodi najmanjših kvadratov $\Leftrightarrow$ Ax^*-b pravokoten na $\operatorname{Im} A$.

Dokaz:

$(\Leftarrow)$: $\quad Ax^*-b \perp \operatorname{Im}A \quad : \underline{\|Ax^*-b\|_2^2 \leq \|Ax-b\|_2^2 \quad \forall x \in \mathbb{R}^n}$

$$x \in \mathbb{R}^n : \|Ax-b\|_2^2 = \|Ax - Ax^* + Ax^* - b\|_2^2 =$$

$$= \|A(x-x^*) + (Ax^*-b)\|_2^2 =$$

$$= \|A(x-x^*)\|_2^2 + \underbrace{2(A(x-x^*))^T(Ax^*-b)}_{0} + \|Ax^*-b\|_2^2 =$$

$$= \|A(x-x^*)\|_2^2 + \|Ax^*-b\|_2^2 \geq \|Ax^*-b\|_2^2$$

$(\Rightarrow)$: $\quad x^* = \operatorname{argmin}\|Ax-b\|_2^2 \quad ; \quad \underline{Ax^*-b \perp \operatorname{Im}A}$

Izberemo poljuben $\lambda > 0$ in poljuben $y \in \operatorname{Im}A$:

$$\|Ax^*-b\|_2^2 \leq \|\underbrace{Ax^*-b+\lambda y}_{\substack{A(x^*+\lambda x) \\ Ax=y}}\|_2^2 = \|Ax^*-b\|_2^2 + 2\lambda y^T(Ax^*-b) + \lambda^2\|y\|_2^2$$

$$2\lambda y^T(Ax^*-b) + \lambda^2\|y\|_2^2 \geq 0 \quad /:\lambda$$

$$2y^T(Ax^*-b) + \lambda\|y\|_2^2 \geq 0$$

$\hookrightarrow$ λ lahko poljubno manjšamo in bo ta del manjši od prvega

$$\Rightarrow \quad y^T(Ax^*-b) \geq 0$$

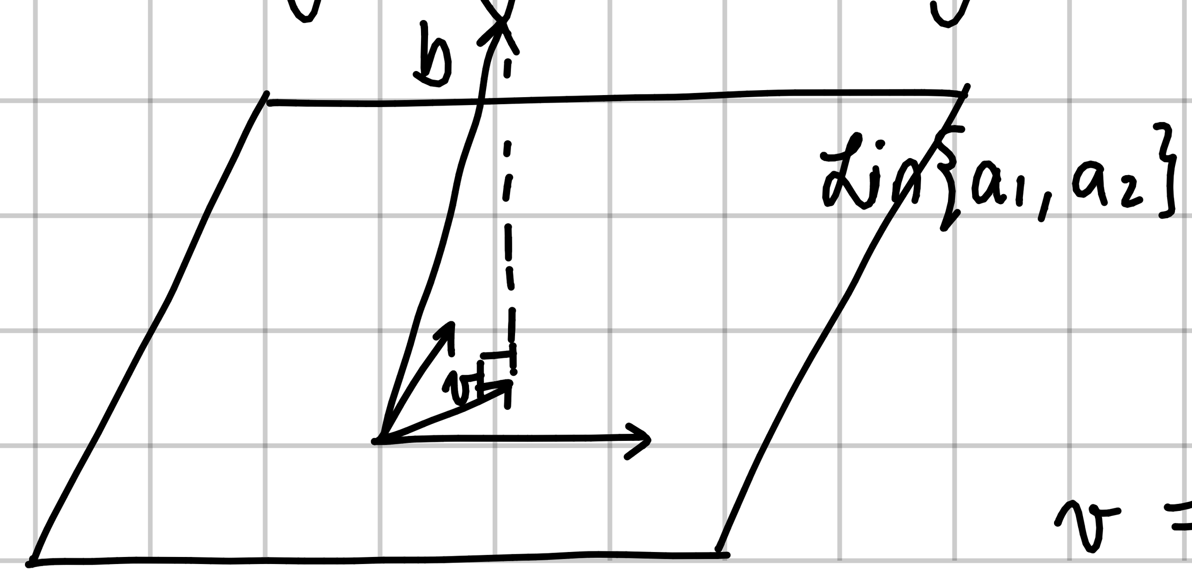
Namesto y izberemo $-y$

$$-2y^T(Ax^*-b) + \lambda\|y\|_2^2 \geq 0$$

kot prej $\Rightarrow$ $\quad -y^T(Ax^*-b) \geq 0 \Rightarrow y^T(Ax^*-b) \leq 0$

Skupaj s prejšnjim je $\quad y^T(Ax^*-b) = 0 \qquad \square$

Primer: Dana sta vektorja $a_1 = [1, 2, -1]^T$ in $a_2 = [1, 1, 0]^T$ ter vektor $b = [1, 1, 1]^T$.
Iščemo vektor v ravnini, ki jo razpenjata a_1 in a_2 tako, da bo najbližje vektorju b.



$$v = \alpha a_1 + \beta a_2$$

$\| v - b \|_2^2$ minimalna

$$F(\alpha, \beta) = \| \alpha a_1 + \beta a_2 - b \|_2^2 \; ; \quad F: \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$\frac{\partial F}{\partial \alpha}(\alpha, \beta) = 0$$

$$\frac{\partial F}{\partial \beta}(\alpha, \beta) = 0$$

$$F(\alpha, \beta) = \| \alpha (1,2,-1)^T + \beta (1,1,0)^T - (1,1,1)^T \|_2^2 =$$

$$= (\alpha + \beta - 1)^2 + (2\alpha + \beta - 1)^2 + (-\alpha - 1)^2$$

$$\frac{\partial F}{\partial \alpha} = 2(\alpha + \beta - 1) + 2(2\alpha + \beta - 1) \cdot 2 + 2(+\alpha + 1)(+1) = 0$$

$$\frac{\partial F}{\partial \beta} = 2(\alpha + \beta - 1) + 2(2\alpha + \beta - 1) = 0$$

$$12\alpha + 6\beta = 4 \quad \rightsquigarrow \quad 6\alpha + 3\beta = 2$$
$$6\alpha + 4\beta = 4 \quad \rightsquigarrow \quad 3\alpha + 2\beta = 2$$

Naj. kvadrati:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ -1 & 0 \end{bmatrix} x = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix} x = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Za normalni sistem je znano:

$$\mathcal{K}(A^T A) \doteq \mathcal{K}(A)^2$$

Standardni QR razcep

$\underline{A = QR}$; $Q \in \mathbb{R}^{m \times n}$ z ortonormiranimi stolpci

$R \in \mathbb{R}^{n \times n}$ zgornje trikotna

Recimo, da razcep imamo:

$$A^T A x = A^T b$$
$$(QR)^T (QR) x = (QR)^T b$$
$$(R^T Q^T)(QR) x = (R^T Q^T) b$$
$$R^T \underbrace{(Q^T Q)}_{= I} R x = R^T Q^T b$$

$$R^T R x = R^T Q^T b \quad / \cdot (R^T)^{-1} \text{ z leve}$$
$$R x = Q^T b$$

Razširjeni QR razcep

$\underline{A = QR}$; $Q \in \mathbb{R}^{m \times m}$ ortogonalna
$\qquad\qquad\quad R \in \mathbb{R}^{m \times n}$ kvazi zg. trikotna

$$A^T A x = A^T b$$
$$(QR)^T (QR) x = (QR)^T b$$
$$R^T \underbrace{Q^T Q}_{I} R x = R^T Q^T b$$
$$R^T R x = R^T Q^T b$$

? $\qquad R^T$ nima inverza ✗

$$\|Ax - b\|_2^2 = \| QRx - b\|_2^2 =$$

pomnožimo s Q^T

$$= \| Rx - Q^T b\|_2^2$$

$x \in \mathbb{R}^n \quad \|x\|_2$

$Qx \in \mathbb{R}^n$, Q ortog. $\quad \|Qx\|_2$

$$\|Qx\| = \|x\|$$
$$\|x\|_2^2 = x^T x$$
$$\|Qx\|_2^2 = (Qx)^T (Qx) = $$
$$= x^T \underbrace{Q^T Q}_{I} x = x^T x$$

$$R = \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix} \qquad Q^T b = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\left\| \begin{bmatrix} \tilde{R} x \\ 0 \end{bmatrix} - \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right\|_2^2 = \left\| \begin{matrix} \tilde{R}x - c_1 \\ -c_2 \end{matrix} \right\|_2^2 = \|\tilde{R}x - c_1\|_2^2 + \|-c_2\|_2^2$$

$$\min_x \|Ax - b\|_2^2 = \min_x \big(\underbrace{\|\tilde{R}x - c_1\|_2^2}_{0} + \underbrace{\|-c_2\|_2^2}_{} \big)$$

$$\Rightarrow \tilde{R} x = c_1$$

najmanjša možna dolžina tistega pravokotnega vektorja

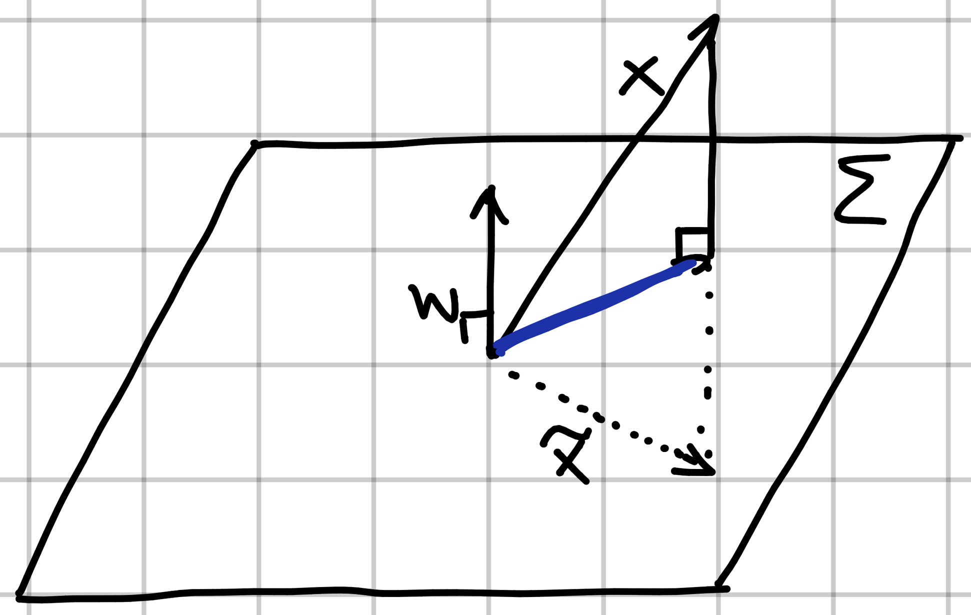
Algoritem za standardni QR razcep

$$A = QR$$

$$\begin{bmatrix} a_1, \dots, a_n \end{bmatrix} = \begin{bmatrix} q_1 \dots q_n \end{bmatrix} \begin{bmatrix} r_{11} & & r_{1n} \\ 0 & \ddots & \\ & & r_{nn} \end{bmatrix}$$

$$a_1 = r_{11} \cdot q_1$$
$$a_2 = r_{12} \cdot q_1 + r_{22} q_2$$
$$\vdots$$

→ Householderjeva zrcaljenja



Σ je hiperravnina v $\mathbb{R}^m$

$w \neq 0$

$\tilde{x}$ je zrcalna slika vektorja x glede na Σ

$$\tilde{x} = x - 2\alpha w$$

$$x - \alpha w \perp w \iff w^T(x - \alpha w) = 0$$

$$\alpha w^T w = w^T x$$

$$\alpha = \frac{w^T x}{w^T w}$$

$$\tilde{x} = x - 2 \cdot \frac{1}{w^T w} \cdot (w^T x) w =$$

$$= x - \frac{2}{w^T w} w (w^T x) =$$

$$= x - \frac{2}{w^T w} (w w^T) x =$$

$$= \underbrace{\left(I - \frac{2}{w^T w} w w^T \right)}_{P} x$$

P ... Householderjevo zrcaljenje

Lastnosti:

1) $P = P^T$
2) P je ortogonalna matrika

$$PP^T = PP = P^2 = I$$

↳ če 2x prezrcališ, moraš priti na isto mesto

Zakaj je to zrcaljenje?

$\det P = -1$

$\det P = \prod_{i=1}^{n} \lambda_i$; λ_i l. vrednosti

L. vektorji:

- $$Pw = \left(I - \frac{2}{w^T w} w w^T\right) w =$$

$$= w - \frac{2}{\cancel{w^T w}} w \cancel{(w^T w)} = -w$$

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$\vdots$

$$a_k = \sum_{j=1}^{k} r_{jk} q_j$$

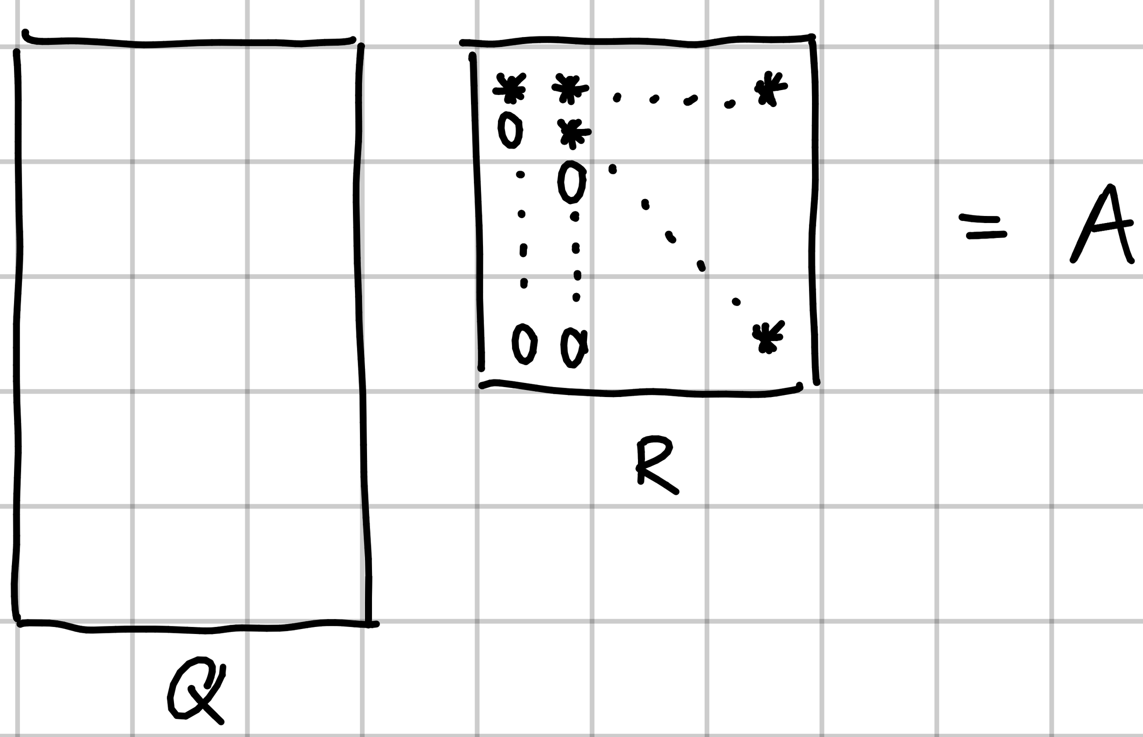
Gram-Schmidt

$\downarrow$

iz nekega nabora vektorjev, naredi ortogonalne vektorje, ki razpenjajo isti prostor

- **standardni** (uporabljamo vseskozi vektorje iz A)
- **modificirani** (uporabimo tiste vektorje, ki smo jih že modificirali)

Bolje je na začetku s svežimi podatki čim več izračunati kot na koncu (zaradi napak).

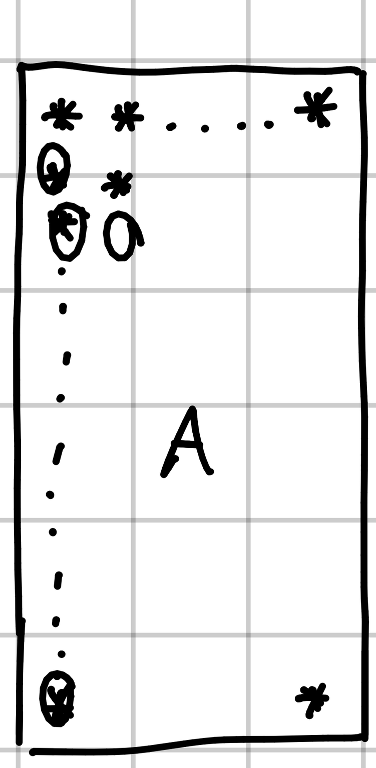


Razširjeni QR razcep

$A \in \mathbb{R}^{m \times n}$, $Q \in \mathbb{R}^{m \times m}$, $R \in \mathbb{R}^{m \times n}$

1.) Givensove rotacije
2.) Householderjeva zrcaljenja

$\rightarrow$ Givensove rotacije

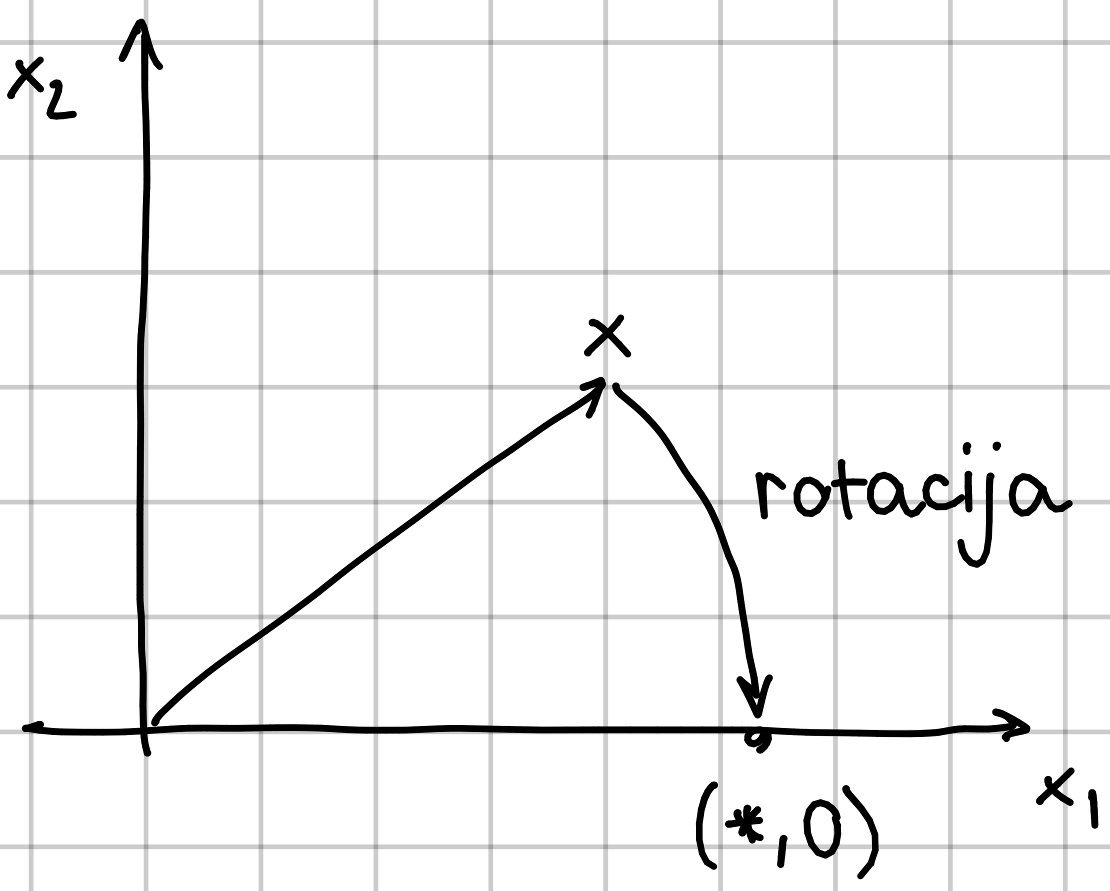


$$R_{12}^T A = \begin{bmatrix} * & \\ 0 & \\ * & \\ \vdots & * \\ * & \end{bmatrix} \quad ; \quad R_{13}^T R_{12}^T A = \begin{bmatrix} * & \\ 0 & \\ 0 & \\ * & * \\ \vdots & \\ * & \end{bmatrix}$$

$\dots$

Rotacija v $\mathbb{R}^2$

$$x \in \mathbb{R}^2 \; ; \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longrightarrow \tilde{x} = \begin{bmatrix} * \\ 0 \end{bmatrix}$$



$$R_{12}^T = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \qquad \begin{array}{l} c = \cos\varphi \\ s = \sin\varphi \end{array}$$

$$R_{ij}^T = \begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & \ddots & & & & & \\ & & & c & & s & & \\ & & & & 1 \cdots 1 & & & \\ & & & -s & & c & & \\ & & & & & & 1 & \\ & & & & & & & \ddots \; 1 \end{bmatrix} \quad \text{(ostalo so 0)}$$

→ spremenita se samo i-ta in j-ta vrstica

Postopek:

$$\underbrace{(R_{nm}^T \cdots R_{13}^T R_{12}^T)}_{Q^T \text{ ... ortogonalna}} A = R = \begin{bmatrix} * & * & & \cdots & * \\ 0 & * & & & \vdots \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & & & * \\ & & & & 0 \\ & & & & \vdots \\ 0 & 0 & \cdots & & 0 \end{bmatrix}$$

$$Q^T A = R \qquad / \cdot Q \text{ z leve}$$
$$A = QR$$

$$Q = R_{12} R_{13} \cdots R_{nm}$$

$(-1, w)$ lastni par

- Pv ; $v \in \Sigma$

$$Iv - \frac{2}{w^T w} w (\underbrace{w^T v}_{0}) = v$$

$\Rightarrow$ l. par: $(1, v)$

$\{v_1, \dots, v_{m-1}\}$ baza $\Sigma \Rightarrow Pv_i = v_i$, $i \in [m-1]$

$\Rightarrow (1, v_i) \Rightarrow \det P = 1 \cdot 1 \cdots 1 \cdot (-1) = -1$

QR razcep:

$$P_1 \begin{bmatrix} * & \\ * & \\ \vdots & A \\ * & \end{bmatrix} = \begin{bmatrix} * & \\ 0 & \\ \vdots & * \\ 0 & \end{bmatrix}$$

$a \in \mathbb{R}^m$;

$$w = \begin{bmatrix} a_1 \pm \|a\|_2 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \Rightarrow Pa = k \cdot e_1$$

Kaj izbrati „numerično"?
→ želimo, da je w čim večji
⇒ $\operatorname{sign}(a_1)$

Primer: $a = [1 \;\; 2 \;\; 2]^T$; $\|a\|_2 = 3$

$$w = \begin{bmatrix} 1+3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$P \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \left(I - \frac{2}{24} (ww^T)\right) \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \frac{1}{12} w \cdot 12 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

Postopek za QR razcep (s Househ. zrc.)

$$A = A_1 = [a_1\ a_2 \ldots a_n] \quad \text{stolpci}$$

$$\downarrow$$
$$w_1$$
$$\downarrow$$
$$P_1$$

$$\Rightarrow P_1 A = P_1 A_1 = \left[\begin{array}{c|ccc} * & * & \cdots & * \\ \hline 0 & * & & \\ \vdots & \vdots & A_2 & \\ 0 & * & & \end{array}\right]$$

2. korak:

$$\tilde{P}_2 A_2 = \begin{bmatrix} * & * & \cdots & * \\ 0 & & & \\ \vdots & & A_3 & \\ 0 & & & \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \tilde{P}_2 & \\ 0 & & & \end{bmatrix}$$

$$\underbrace{(P_n \cdots P_2 P_1)}_{Q^T} A = R$$

$$A = QR$$

$$Q = P_1 P_2 \cdots P_n$$

↳ ker so simetrične

če rešujemo kvadratni sistem
→ LU hitrejši
→ QR natančnejši

Primer: S pomočjo QR razcepa rešite sistem.

$$2x + 2y + 6z = 6$$
$$2x + y - 2z = -1$$
$$x + 6y - 2z = -7$$

$$A = \begin{bmatrix} 2 & 2 & 6 \\ 2 & 1 & -2 \\ 1 & 6 & -2 \end{bmatrix}; \quad b = \begin{bmatrix} 6 \\ -1 \\ -7 \end{bmatrix}$$

norma od $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$

$$P_1 = I - \frac{2}{w^T w} w w^T \quad ; \quad w_1 = \begin{bmatrix} 2+3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$$

$$P_1 A = [P_1 a_1 \quad P_1 a_2 \quad P_1 a_3]$$

$$\left(I - \frac{2}{w_1^T w_1} w_1 w_1^T\right) a_i = a_i - \frac{2}{w_1^T w_1} w_1 (w_1^T a_i) \quad , \quad i = 1,2,3$$

$$i=1: \quad \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - \frac{2}{30} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} 15 = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$$i=2: \quad \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} - \frac{2}{30} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \cdot 18 = \begin{bmatrix} -4 \\ -7/5 \\ 24/5 \end{bmatrix}$$

$$i=3: \quad \begin{bmatrix} 6 \\ -2 \\ -2 \end{bmatrix} - \frac{2}{30} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} 24 = \begin{bmatrix} -2 \\ -26/5 \\ -18/5 \end{bmatrix}$$

$$P_1 b = \begin{bmatrix} 6 \\ -1 \\ -7 \end{bmatrix} - \frac{2}{30} \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} 21 = \begin{bmatrix} -1 \\ -19/5 \\ -42/5 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -4 & -2 \\ 0 & -7/5 & -26/5 \\ 0 & 24/5 & -18/5 \end{bmatrix} x = \begin{bmatrix} -1 \\ -19/5 \\ -42/5 \end{bmatrix}$$

...

$$\begin{bmatrix} -3 & -4 & -2 \\ 0 & 5 & -2 \\ 0 & 0 & -6 \end{bmatrix} x = \begin{bmatrix} -1 \\ -7 \\ -6 \end{bmatrix}$$

$x_3 = 1$
$x_2 = -1$
$x_1 = 1$

9. 1. 2026

Modeliranje širjenja COVID okužbe (začetna faza)

t_i	1	2	3	. . .	10
f_i	1	2	8	12 . . .	141

$$f(t) = a \cdot e^{bt} \quad \ldots \text{ nelinearna funkcija } a, b$$

$$\log f(t) = \underset{\substack{\parallel \\ A}}{\log a} + bt = A + bt \overset{mnk}{\Longrightarrow} A, b$$

$$\Rightarrow \quad a = e^A, \quad b = b$$

SINGULARNI RAZCEP (SVD)

$A \in \mathbb{R}^{n \times n}$ ... lastni par (λ, v)

$$Av = \lambda v, \quad v \neq 0$$

Ali se da A diagonalizirati?

$$A = PDP^{-1};$$

P... prehodna matrika
D... diagonalna

→ V splošnem to ni izvedljivo

Jordanova forma: $A = PJP^{-1}$

$$\begin{bmatrix} \begin{array}{cccc|ccc|c} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & & \\ & & \ddots & 1 & & & & \\ & & & \lambda_1 & & & & \\ \hline & & & & \lambda_2 & 1 & & \\ & & & & & \ddots & 1 & \\ & & & & & & \lambda_2 & \\ & & & & & & & \ddots \\ & & & & & & & \boxed{\begin{matrix} \lambda_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{matrix}} \end{array} \end{bmatrix}$$

$$\ker\left((A - \lambda_i I)^k\right); \quad k = 1, 2, \ldots$$

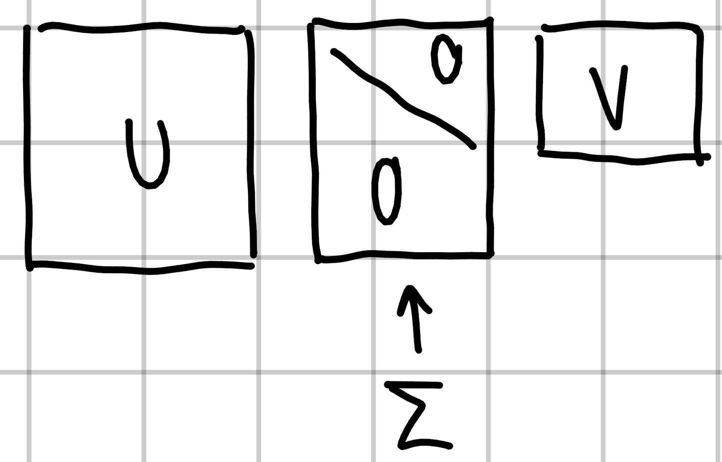
Primer: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ → se ne da diagonalizirati

Singularni razcep:

$A \in \mathbb{R}^{m \times n}$; $m \geq n$

Trdimo, da obstajajo matrike $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$, $\Sigma \in \mathbb{R}^{m \times n}$:

$A = U \Sigma V^T$; U, V ortogonalni
Σ kvazi diagonalna



TRDITEV: Singularni razcep vedno obstaja.
Torej obstajata ortogonalni matriki $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$ ter matrika

$$\Sigma = \begin{bmatrix} \sigma_1 & & & 0 \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ & 0 & & \end{bmatrix} \in \mathbb{R}^{m \times n}$$

tako, da je $A = U \Sigma V^T$, $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$.
↳ pozitivni koreni σ_i^2

Stolpci matrike $U = [u_1, u_2, \ldots, u_m]$ so levi, stolpci matrike $V = [v_1, \ldots, v_n]$ pa desni singularni vektorji.

Dokaz: Pogledamo matriko $A^T A$: simetrična, pozitivno semidefinitna
Torej obstajajo lastne vrednosti $\sigma_i^2 \geq 0$ in ortonormirani vektorji v_i: $A^T A v_i = \sigma_i^2 v_i$; $i = 1, 2, \ldots, n$

Naj bo $\sigma_r > 0$ in $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$; $r \leq n$

Označimo: $V_1 = [v_1, v_2, \ldots, v_r]$, $V_2 = [v_{r+1}, \ldots, v_n]$

$$(A V_2)^T (A V_2) = V_2^T \underbrace{A^T A V_2} = V_2^T \cdot 0 = 0$$
$$\Rightarrow A V_2 = 0$$

Definiramo: $u_i = \frac{1}{\sigma_i} \cdot A v_i$; $i = 1, \ldots, r$

Vektorji u_i so ortonormirani.

$$u_j^T u_i = \left(\frac{1}{\sigma_j} A v_j\right)^T \frac{1}{\sigma_i} A v_i =$$
$$= \frac{1}{\sigma_j \sigma_i} \cdot v_j^T A^T A v_i = \frac{1}{\sigma_j \sigma_i} \cdot v_j^T \cdot \sigma_i^2 \cdot v_i =$$
$$= \frac{\sigma_i^2}{\sigma_j \sigma_i} v_j^T v_i = \begin{cases} 1 ; & i = j \\ 0 ; & i \neq j \end{cases} = \delta_{ij}$$

Definiramo $U_1 = [u_1, u_2, \dots, u_r]$.

Lahko izberemo $U_2 = [u_{r+1}, \dots, u_m]$ tako, da je $U = [U_1 \ U_2]$ ortogonalna.

Matrika $U^T A V = [U_1 \ U_2]^T A [V_1 \ V_2]$ ima obliko

$$U^T A V = \begin{bmatrix} U_1^T A V_1 & U_1^T A V_2 \\ U_2^T A V_1 & U_2^T A V_2 \end{bmatrix}$$

($U_1^T A V_2 = 0$, $U_2^T A V_2 = 0$, $U_2^T A V_1 = 0$)

$$U_2^T A V_1 = [u_{r+1}, \dots, u_m]^T A [v_1, \dots, v_r] = 0$$

$$U_1^T A V_1 = [u_1, \dots, u_r]^T A [v_1, \dots, v_r] = \mathrm{diag}(\sigma_i)_{i=1}^r$$

Torej je

$$U^T A V = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & 0 \ \cdots \ 0 \\ & 0 & & & \end{bmatrix} = \Sigma$$

Lema: Če je $A \in \mathbb{R}^{m\times n}$, $\mathrm{rang}(A) = n$, potem je minimum $\|Ax - b\|_2$ dosežen pri $x = \sum_{i=1}^n \frac{u_i^T b}{\sigma_i} v_i$

Dokaz:

Naj bo $A = U\Sigma V^T$ in $U = [\underset{n}{U_1} \ \underset{m-n}{U_2}]$, $\Sigma = \begin{bmatrix} S \\ 0 \end{bmatrix}\begin{matrix} n \\ m-n \end{matrix}$

$$\|Ax - b\|_2^2 = \|U\Sigma V^T x - b\|_2^2 = \|U^T(U\Sigma V^T x - b)\|_2^2 =$$

$$= \|\Sigma V^T x - U^T b\|_2^2 = \left\| \begin{bmatrix} S \\ 0 \end{bmatrix} V^T x - \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} b \right\|_2^2 =$$

$$= \left\| \begin{bmatrix} S V^T x - U_1^T b \\ 0 - U_2^T b \end{bmatrix} \right\|_2^2 \Rightarrow S V^T x = U^T b$$

$$x = V S^{-1} U_1^T b$$

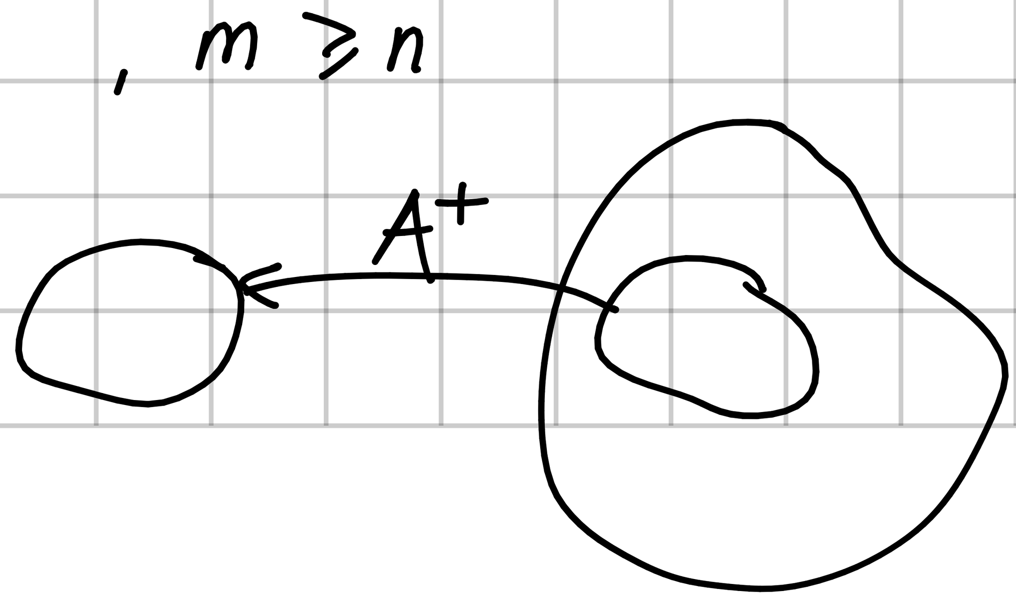
$$x = \sum_{i=1}^n \frac{u_i^T b}{\sigma_i} v_i$$

$\square$

PSEVDOINVERZ (Moore-Penrose)

$A \in \mathbb{R}^{m\times n}$, $m \geq n$

$A: \mathbb{R}^n \to \mathbb{R}^m$



$A^+ \dots$ psevdoinverz

DEF.: Za matriko $A \in \mathbb{R}^{m\times n}$, $m \geq n$, $\operatorname{rang}(A) = n$, definiramo psevdoinverz $A^+ \in \mathbb{R}^{n\times m}$ kot
$$A^+ = (A^TA)^{-1}A^T .$$

V primeru ko je $m < n$ in $\operatorname{rang}(A) = m$, je $A^+ = A^T(AA^T)^{-1}$.

DEF.: Matrika $X \in \mathbb{R}^{n\times m}$ je psevdoinverz matrike $A \in \mathbb{R}^{m\times n}$, če izpolnjuje Moore-Penroseove pogoje:
1) $AXA = A$
2) $XAX = X$
3) $(AX)^T = AX$
4) $(XA)^T = XA$

IZREK: Naj bo $A \in \mathbb{R}^{m\times n}$, $m \geq n$, $\operatorname{rang}(A) = r$ in $A = U\Sigma V^T$, kjer je $U = [\underset{r}{U_1} \ \underset{m-r}{U_2}]$, $V = [\underset{r}{V_1} \ \underset{n-r}{V_2}]$, $\Sigma = \begin{array}{c|c} & \begin{matrix} r & n-r \end{matrix} \\ \begin{matrix} r \\ m-r \end{matrix} & \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \end{array}$, $S = \operatorname{diag}(\sigma_i)_{i=1}^r$.
Psevdoinverz matrike A je enoličen in je enak
$$A^+ = V\Sigma^+U^T \quad , \text{ kjer je } \quad \Sigma^+ = \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} .$$

Dokaz: Psevdoinverz A^+ je velikosti $n\times m$, zato ga lahko pišemo kot $A^+ = VYU^T$, kjer je
$$Y = \begin{array}{c|c} \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} & \begin{matrix} r \\ m-r \end{matrix} \\ \begin{matrix} r & n-r \end{matrix} & \end{array}$$

Pogledamo produkt $AA^+ = (U\Sigma V^T)(VYU^T) =$
$$= U \begin{bmatrix} SY_{11} & SY_{12} \\ 0 & 0 \end{bmatrix} U^T$$

Po 3) je $AA^+ = (AA^+)^T$, od koder sledi $Y_{12} = 0$.
Podobno iz 4) dobimo, da je $Y_{21} = 0$.

Sedaj iz 2) sledi $A^+AA^+ = A^+$, torej je
$$\begin{bmatrix} Y_{11}SY_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{22} \end{bmatrix} .$$
Torej je $Y_{22} = 0$.

Iz lastnosti 1) dobimo $AA^+A = A$, torej $SY_{11}S = S \quad / \cdot S^{-1}$ z leve in desne

$$\Rightarrow \quad Y_{11} = S^{-1} .$$ $\square$

16. 1. 2026

IZREK (Echart - Young - Mirsky):

$m \geq n$

Naj bo $A = U\Sigma V^T$, $A \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{m\times m}$ in $V \in \mathbb{R}^{n\times n}$ ortogonalni in $\Sigma \in \mathbb{R}^{m\times n}$ kvazi diagonalna s singularnimi vrednostmi $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$. Naj bo $\text{rang}(A) > k$, $0 < k < n-1$.
Naj bo $A_k = \sum_{i=1}^{k} \sigma_i \cdot u_i v_i^T$, potem je

$$\min_{\text{rang}(B)=k} \|B-A\|_2 = \|A_k - A\|_2 = \sigma_{k+1}.$$

$$\min_{\text{rang}(B)=k} \|B-A\|_F = \|A_k - A\|_F = \sqrt{\sum_{i=k+1}^{n} \sigma_i^2}$$

Dokaz: Naj bo $B \in \mathbb{R}^{m\times n}$ matrika ranga k.
$\dim \ker B = n-k$ $(n - \text{rang})$

Definiramo $V_{k+1} = [\underbrace{v_1, v_2, \ldots, v_{k+1}}_{\in V}]$, $\dim \operatorname{Im} V_{k+1} = k+1$

$$\dim \ker B + \dim \operatorname{Im} V_{k+1} = n-k+k+1 = n+1$$
$$\Rightarrow \exists z \neq 0;\ z \in \ker B \cap \operatorname{Im} V_{k+1};\ \|z\|_2 = 1$$

$$\|B-A\|_2 \geq \|(B-A)z\|_2 = \|Az\|_2 = \|U\Sigma V^T z\|_2 =$$

po def. 2. norme

$$= \|\Sigma V^T z\|_2 = \left\| \begin{bmatrix} S \\ 0 \end{bmatrix} [V_{k+1}, \tilde{V}]^T z \right\|_2 = \left\| \begin{bmatrix} S \\ 0 \end{bmatrix} \begin{bmatrix} V_{k+1}^T \\ \tilde{V}^T \end{bmatrix} z \right\|_2 =$$

$$= \left\| \begin{bmatrix} S \\ 0 \end{bmatrix} \begin{bmatrix} V_{k+1}^T z \\ 0 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} S \\ 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\|_2 =$$

$z = \sum_{j=1}^{k+1} \alpha_j v_j$, $\sum_{j=1}^{k+1} \alpha_j^2 = 1$

$$= \sqrt{\sum_{j=1}^{k+1} (\sigma_j \alpha_j)^2} \geq \sqrt{\sigma_{k+1}^2 \cdot \sum_{j=1}^{k+1} \alpha_j^2} =$$
$$= \sigma_{k+1} \cdot 1 = \sigma_{k+1}$$

Izberemo $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$. Trdimo, da je $\text{rang}(A_k) = k$ (DN)

Če za B izberemo $A_k \Rightarrow \|A_k - A\| \geq \sigma_{k+1}$.

Vzemimo $z = v_{k+1}$: $\|(A_k - A)z\|_2 = \left\|\left(\sum_{i=1}^{k} \sigma_i u_i v_i^T - \sum_{i=1}^{n} \sigma_i u_i v_i^T\right) z\right\|_2 =$

$$= \left\|\left(\sum_{i=k+1}^{n} \sigma_i u_i v_i^T\right) v_{k+1}\right\|_2 = \|\sigma_{k+1} u_{k+1} \cdot 1\|_2 = \sigma_{k+1} \cdot 1 = \sigma_{k+1}$$

$$\cancel{\Rightarrow \min_{\text{rang}(B)=k} \|B-A\|_2 = \sigma_{k+1}}$$

Izberemo $z \in \mathbb{R}^n$, $\|z\|_2 = 1$. Torej je $z = \sum_{j=1}^{n} \alpha_j v_j$, $\sum_{j=1}^{n} \alpha_j^2 = 1$

$$\|(A_k - A)z\|_2 = \left\|\left(\sum_{i=k+1}^{n} \sigma_i u_i v_i^T\right) \cdot \sum_{j=1}^{n} \alpha_j v_j\right\|_2 = \left\|\sum_{j=1}^{n} \alpha_j \left(\sum_{i=k+1}^{n} \sigma_i u_i v_i^T v_j\right)\right\|_2 =$$

$$= \left\| \sum_{j=k+1}^{n} \alpha_j \sigma_j u_j \cdot 1 \right\|_2 = \sqrt{\sum_{j=k+1}^{n} \alpha_j^2 \sigma_j^2} \leq \sqrt{\sum_{j=k+1}^{n} \alpha_j^2 \sigma_{k+1}^2} = \sigma_{k+1} \sqrt{\sum_{j=k+1}^{n} \alpha_j} \leq \sigma_{k+1}$$

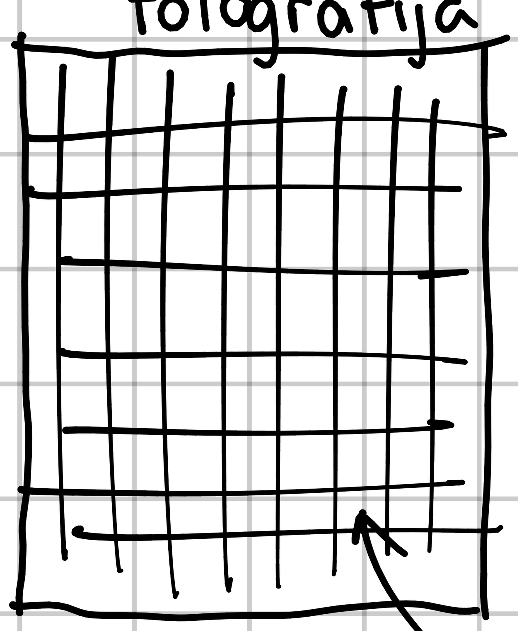
$$\Rightarrow \sup_{\|z\|_2=1} \|(A_k - A)z\|_2 = \|A_k - A\|_2 \leq \sigma_{k+1}$$

$\square$

Uporaba pri stiskanju slik:

fotografija

m



$\in \mathbb{R}^{m\times n}$, $m \geq n$

n $\quad a_{ij} \in \{0, \dots, 255\}$

$\text{rang}(A) = n$ ali pa blizu temu

Iščemo aproksimacijo A_k, ki zahteva manj podatkov.

$$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T \qquad \text{(SVD)}$$

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \qquad k \ll n$$

Predoločeni sistemi z defektnim rangom

$A \in \mathbb{R}^{m\times n}$, $m \geq n$, $\text{rang}(A) = r < n$

TRDITEV: Če je $A \in \mathbb{R}^{m\times n}$, $m \geq n$, $\text{rang}(A) = r < n$ in $A = U\Sigma V^T$ singularni razcep, potem ima med vsemi vektorji, ki minimizirajo $\|Ax-b\|_2$, najmanjšo normo $x = A^+b$, oziroma $x = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$, $\|Ax-b\|_2^2 = \sum_{i=r+1}^{n} (u_i^T b)^2$.

Dokaz: $x \in \mathbb{R}^n$: $\|Ax-b\|_2^2 = \|U\Sigma V^T x - b\|_2^2 = \|\Sigma \underbrace{V^T x}_{a} - U^T b\|_2^2 =$

$$= \sum_{i=1}^{r} (\sigma_i a_i - u_i^T b)^2 + \sum_{i=r+1}^{n} (u_i^T b)^2$$

Minimum bo dosežen, če je $\sigma_i a_i - u_i^T b = 0$ za $i \in [r]$

$$\Rightarrow a_i = \frac{u_i^T b}{\sigma_i}$$

Ker je $V^T x = a \Rightarrow \quad x = Va = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i + \sum_{i=r+1}^{n} a_i v_i$

$$\|x\|_2^2 = \|Va\|_2^2 = \|a\|_2^2$$

$$= \sum_{i=1}^{n} a_i^2$$

$$= \sum_{i=1}^{r} a_i^2 + \sum_{i=r+1}^{n} a_i^2$$

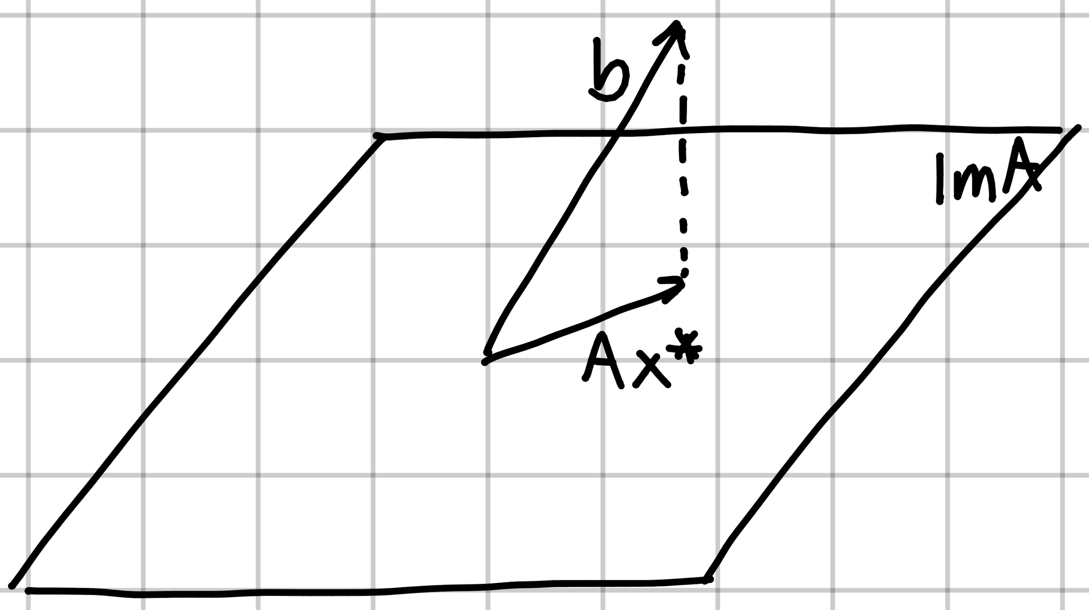
Min. je dosežen, če je $a_{r+1} = \dots = a_n = 0$

$\square$

Problem totalnih najmanjših kvadratov

Klasičen problem najmanjših kvadratov: $\min_x \|Ax - b\|_2$

$\text{rang}(A) = n$



Če ostanek označimo z $r = Ax - b \Rightarrow Ax = b + r \in \text{Im}A$

Iščemo torej $\tilde{b} \in \text{Im}A$: $\|\tilde{b} - b\|_2$ je minimalna.

Če so napake tudi v matriki A, potem je smiselno poiskati $\tilde{A}$ in $\tilde{b} \in \text{Im}\tilde{A}$, tako, da je $\|[\tilde{A}, \tilde{b}] - [A, b]\|_F$.

$\text{rang}(\tilde{A}) = n$

Rešitev nam da singularni razcep $[A, b]$.

Recimo, da $b \notin \text{Im}A \Rightarrow \text{rang}[A, b] = n+1$

Iz $\tilde{b} \in \text{Im}\tilde{A}$ sledi $\text{rang}[\tilde{A}, \tilde{b}] = n$.

Naj bo $[A, b] = U \Sigma V^T$, $\sigma_n \geq \sigma_{n+1} > 0$.

Izrek E-Y-M:

Matrika ranga n, ki v Frobeniusovi normi najbolje aproksimira $[A, b]$ je $[\tilde{A}, \tilde{b}] = [A, b] - \sigma_{n+1} u_{n+1} v_{n+1}^T$

Ker je $[\tilde{A}, \tilde{b}] v_{n+1} = (U \Sigma V^T) v_{n+1} - \sigma_{n+1} u_{n+1} \underbrace{v_{n+1}^T v_{n+1}}_{1} =$

$$= (U \Sigma (V^T) v_{n+1}) - \sigma_{n+1} u_{n+1} =$$

$$= \sigma_{n+1} u_{n+1} \cdot 1 - \sigma_{n+1} u_{n+1} = 0$$

Torej je $v_{n+1} \in \text{Ker}[\tilde{A}, \tilde{b}] \Rightarrow \text{Ker}[\tilde{A}, \tilde{b}] = \mathcal{L}in\{v_{n+1}\}$

Če je $\tilde{x}$ rešitev $\tilde{A}\tilde{x} = \tilde{b}$, je $[\tilde{A} \ \tilde{b}] \begin{bmatrix} \tilde{x} \\ -1 \end{bmatrix} = 0$

$$\Rightarrow \begin{bmatrix} \tilde{x} \\ -1 \end{bmatrix} \in \operatorname{Ker}[\tilde{A}, \tilde{b}] = \mathcal{L}in\{v_{n+1}\}$$

$$\begin{bmatrix} \tilde{x} \\ -1 \end{bmatrix} = \alpha \cdot v_{n+1}$$

$$-1 = \alpha \cdot (v_{n+1})_{n+1} \Rightarrow \alpha$$

$$\tilde{x} = \begin{bmatrix} v_{n+1,1} \\ \vdots \\ v_{n+1,n} \end{bmatrix}$$

Primer: Iščemo premico $y = kx$, ki najbolje aproksimira podatke

x	−2	−1	0	1	2
y	1.5	0.2	0.5	−2.3	−1.5

a) po klasični metodi najmanjših kvadratov

b) po metodi totalnih min. kvadratov

$$A_1 = [A, b], \quad A = \begin{bmatrix} x_1 \\ \vdots \\ x_5 \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$A_1 = U \Sigma V^T$$

V_2 zadnji stolpec

$$\alpha = -1 / v_{2,2}$$

$k =$